An Algorithm for Nonconvex Functional Minimization and Applications to Image Restoration

Bartomeu Coll, Joan Duran and Catalina Sbert
tomeu.coll@uib.es, joan.duran@uib.es, catalina.sbert@uib.es
Department of Mathematics and Computer Science
University of Balearic Islands

Govern de les Illes Balears
Conselleria d'Educació, Cultura i Universitats
Direcció General d'Universitats, Recerca i Transferència del Coneixement
Invertim en el seu futur
Unió Europea
Fons Social Europeu

Abstract

We propose a dual algorithm for the minimization of discrete nonconvex functionals. Our approach consists in introducing an auxiliary variable, with explicit expression, that marks the edges and makes the regularization *half-linear* in the sense that the dual energy involves a weighted total variation with respect to the main variable. Experimental results show that, whatever may be the perturbations to which an image is subjected, the proposed algorithm provides piecewise constant restored images with neat edges.

1. Introduction

Image restoration consists in the reconstruction of an original image from a degraded observation of it. In the variational framework, this leads to the minimization of a functional involving a data-fidelity term and a regularization term [5]. Nonconvex energies have become a standard tool in image processing. Although numerical tests revealed that nonconvexity gives better results than convexity, theoretical deficiencies on the existence of solution and numerical intricacies restrict the algorithms that can be used.

We are interested in expanding the scope of edge-preserving convex approximations to nonsmooth nonconvex TV-based regularization. Our proposal is related with the *half-quadratic* criterion [3], which provides a deterministic algorithm minimizing a dual Laplacian-like functional, and with the weighted total variation [1].

2. The Discrete Minimization Problem

Let us denote an image as a two-dimensional matrix in the euclidean space $X := \mathbb{R}^{N\times N}$. If $u \in X$, the discrete gradient is a vector in $Y = X \times X$ given at each pixel by $(\nabla u)_{i,j} = \left((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j}\right)$, with

$$(\nabla u)^1_{i,j} = \begin{cases} \dfrac{u_{i+1,j} - u_{i,j}}{\alpha} & \text{if } i < N, \\ 0 & \text{if } i = N, \end{cases} \quad \text{and} \quad (\nabla u)^2_{i,j} = \begin{cases} \dfrac{u_{i,j+1} - u_{i,j}}{\alpha} & \text{if } j < N, \\ 0 & \text{if } j = N, \end{cases}$$

where $\alpha > 0$ is a scaling parameter that tunes the value of the gradient above which a discontinuity is detected.

We are interested in the minimization of the discrete objective functional

$$\min_{u\in X} J(u) := \sum_{1\le i,j\le N} \phi\left(\left|(\nabla u)_{i,j}\right|\right) + \frac{\lambda}{2} \sum_{1\le i,j\le N} \left((Au)_{i,j} - f_{i,j}\right)^2. \tag{1}$$

In this setting, $f \in X$ is the degraded data, $u \in X$ is the underlying image to be recovered, A is the linear operator modeling the degradation of u, ϕ is a so-called *potential function*, and $\lambda \ge 0$ is a trade-off parameter.

3. Half-linear Regularization

Some **requirements** are imposed to have forward diffusion (**smoothing**) **in quasi-constant areas** and backward diffusion (**enhancement**) **around edges**:

- **(A1)** $\phi : [0, +\infty) \to [0, +\infty)$ strictly increases on $(0, +\infty)$, with $\phi(0) = 0$.
- **(A2)** ϕ is twice continuously differentiable on $(0, +\infty)$.
- **(A3)** ϕ' strictly decreases on $(0, +\infty)$.
- **(A4)** $\lim_{t\to+\infty} \phi'(t) = 0$.
- **(A5)** $\lim_{t\downarrow 0^+} \phi'(t) = M$, with $0 < M < +\infty$.

Formally, from the Euler-Lagrange equation

$$-\text{div}\left(\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) + \lambda A^*(Au - f) = 0,$$

a qualitative reasoning on the above conditions in terms of a weighted TV regularizer is developed:

- ϕ' acts as a weight for the total variation or, equivalently, for the mean curvature.
- (A4) implies that the Euler-Lagrange equation behaves as the usual TV at pixels in an homogeneous area.
- (A5) penalizes the diffusion in the gradient direction at pixels close to edges and, thus, edges are preserved.

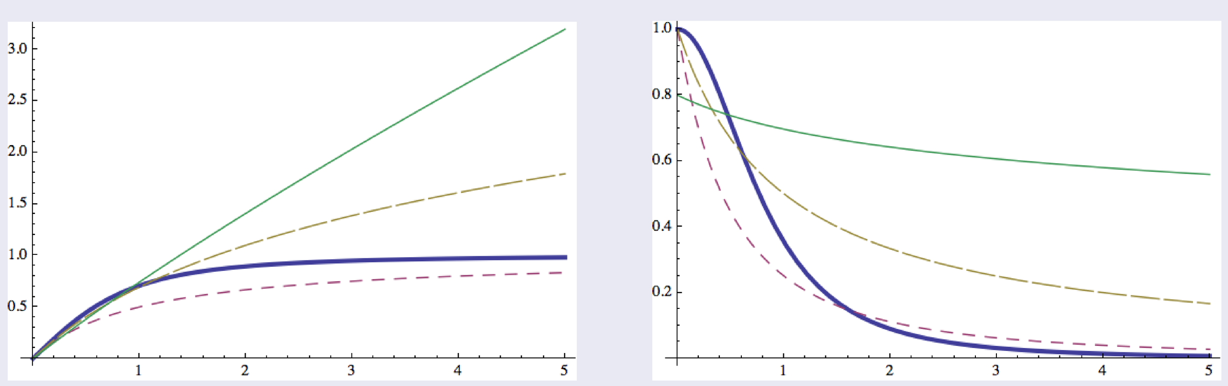
Theorem (Main dual theorem)

Let ϕ satisfy (A1) – (A5), then there exists a strictly convex and strictly decreasing function $\psi : (0, M] \to [0, \beta)$, with $\beta = \lim_{t\to+\infty}(\phi(t) - t\phi'(t))$, such that

$$\phi(t) = \min_{\omega\in(0,M]} (\omega t + \psi(\omega)) = t\phi'(t) + (\psi \circ \phi')(t), \quad \forall t \ge 0. \tag{2}$$

Some examples of potential functions satisfying (A1)–(A5) are

$$\phi_1(t) = \frac{t}{\sqrt{1+t^2}}, \quad \phi_2(t) = \frac{t}{1+t}, \quad \phi_3(t) = \ln(1+t), \quad \phi_4(t) = (1+t)^\gamma, \quad \gamma \in (0,1).$$



$\phi(t)$ vs $t = |\nabla u|$ $\qquad$ $\phi'(t)$ vs $t = |\nabla u|$

4. The Proposed Dual Algorithm

In view of (2), we introduce the following **augmented energy** by means of a **dual variable** ω:

$$J^*(u,\omega) = \sum_{1\le i,j\le N} \left(\omega_{i,j}\left|(\nabla u)_{i,j}\right| + \psi\left(\omega_{i,j}\right)\right) + \frac{\lambda}{2}\sum_{1\le i,j\le N}\left((Au)_{i,j} - f_{i,j}\right)^2,$$

where $\omega = (\omega_{i,j})_{1\le i,j\le N}$, with $\omega_{i,j} \in (0, M]$ for any $i, j \in \{1, \dots N\}$. Then, the primal problem (1) can be rewritten as

$$\min_{u\in X} J(u) = \min_{u\in X}\left\{\min_{\omega\in(0,M]^{N\times N}} J^*(u,\omega)\right\} = \min_{u\in X} J^*\left(u, \phi'(|\nabla u|)\right).$$

We propose the following **dual algorithm** that alternates minimizations over u and ω:

$u^0 = f$
repeat
 $\omega^{n+1} = \phi'(|\nabla u^n|)$
 $u^{n+1} = \arg\min_{u\in X} J^*\left(u, \omega^{n+1}\right)$
until convergence

The convex problem $\min_{u\in X} J^*(u, \omega^{n+1})$ can be solved, for instance, by Chambolle's projection algorithm [2]. We proved in [4] the convergence of the proposed approach.

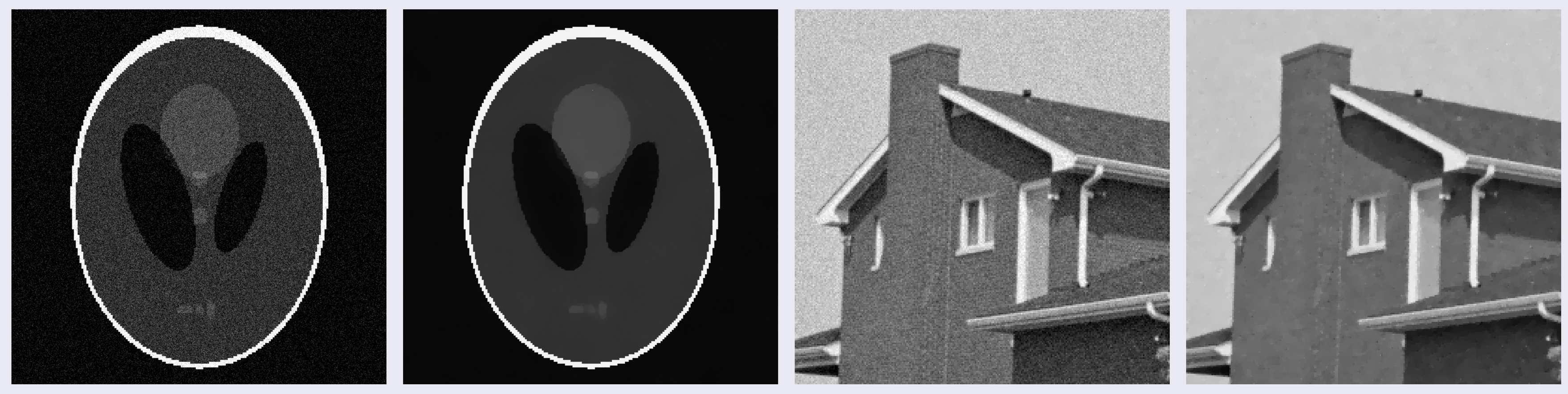
5. Experimental Results

The role of the weight ω as edge detector

Clean data | $\alpha = 20$ | $\alpha = 50$ | Noisy data | $\alpha = 30$

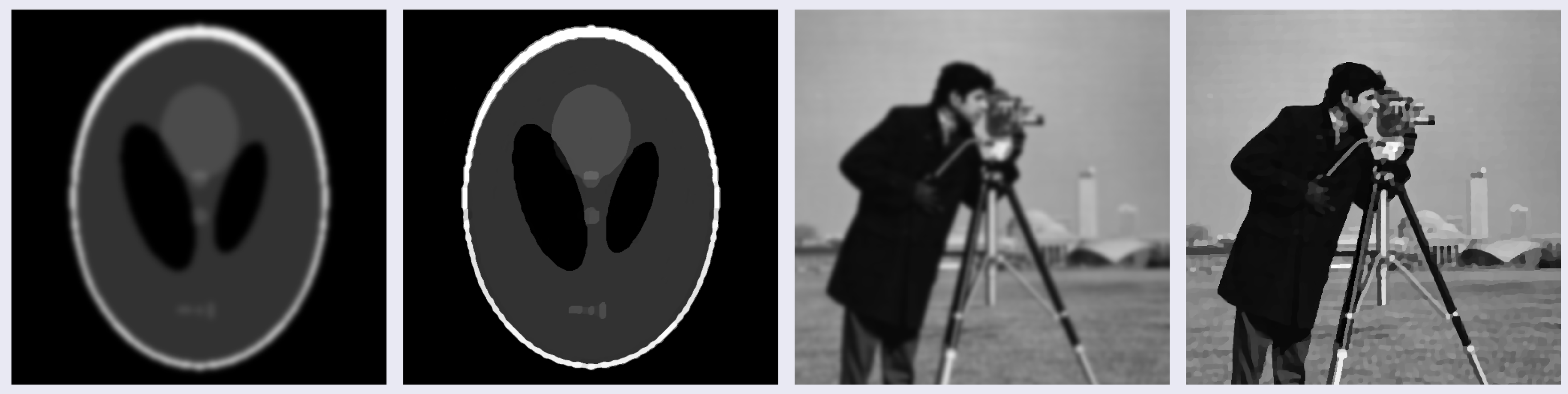
Figure : Weights associated to the final restored images obtained from clean Book image and from noisy Phantom image (noise s.d. $\sigma = 10$). The dual variable perfectly detects image features and it provides neat edges. As α increases, more gradients are penalized.

Applications of the proposed algorithm in image restoration



PSNR = 21.97 | PSNR = 30.44 | PSNR = 28.14 | PSNR = 33.69

Figure : Denoising experiment on piecewise constant Phantom image corrupted with noise s.d. $\sigma = 25$ and on textured House image corrupted with noise s.d. $\sigma = 10$. In all cases, the algorithm provides piecewise constant images with neat edges and closed regions.



PSNR = 20.27 | PSNR = 24.87 | PSNR = 24.64 | PSNR = 25.01

Figure : Deconvolution experiment on piecewise constant Phantom image convolved with Gaussian kernel s.d. $\sigma = 5$ and on Cameraman image convolved with Gaussian kernel s.d. $\sigma = 3$. Once more, the algorithm provides segmented-restored solutions from blurred data.

Performance comparison on image denoising

Image	Noisy image		Chambolle		Split Bregman		Half-quadratic		Ours	
	RMSE	PSNR	RMSE	PSNR	RMSE	PSNR	RMSE	PSNR	RMSE	PSNR
Book	17.38	24.65	7.21	31.50	8.09	30.44	7.76	31.06	**6.99**	**31.88**
Cameraman	16.81	24.91	9.00	29.64	10.04	28.68	8.92	29.91	**8.26**	**30.46**
Phantom	14.24	26.41	7.20	32.33	9.22	30.03	5.77	34.21	**5.29**	**35.31**
Avg.	16.14	25.32	7.80	31.16	9.12	29.72	7.48	31.73	**6.85**	**32.55**

Table : For each image, the averages of the RMSE and the PSNR values over s.d. $\sigma \in \{5, 10, 15, 20, 25, 30\}$ are displayed.

Noisy image | Chambolle | Split Bregman | Half-quadratic | Ours

Figure : Denoising experiment on Phantom image corrupted with noise s.d. $\sigma = 30$. All results except ours suffer from over-smoothing and, furthermore, some image features have disappeared. On the contrary, our result looks clear and best approaches the ground truth.

Extension to color images

Ground truth | Noisy (PSNR = 24.62) | Chambolle (PSNR = 31.73) | Ours (PSNR = 32.48)

Figure : Denoising of color image corrupted with noise s.d. $\sigma = 15$. Our result better preserves edges and it looks clearer. Indeed, the letters from the cover book are blurred in the Chambolle's result.

6. Conclusions

- We have introduced the *half-linear regularization* for dealing with the minimization of discrete nonsmooth nonconvex energy functionals.
- The proposed approach introduces a closed-form dual variable that correctly detects edges and preserves them from smoothing.
- We have proposed a dual algorithm that consists in alternating minimization over the primal and the dual variables.
- The experimental results have shown that our method provides piecewise constant images with neat edges from arbitrary noisy and blurred data.
- Extensions and further developments of our approach have been considered in [4].

Acknowledgements

The authors were supported by Ministerio de Ciencia e Innovación under grant TIN2011-27539. J. Duran had a fellowship of the Conselleria d'Educació, Cultura i Universitats of the Govern de les Illes Balears for the realization of his Ph.D. thesis, which was selected under an operational program co-financed by the European Social Fund.

References

[1] G. Aubert, A. El Hamidi, C. Ghannam, and M. Ménard.
On a class of ill-posed minimization problems in image processing.
J. Math. Anal. Appl., 352(1):380–399, 2009.

[2] A. Chambolle.
An algorithm for total variation minimization and applications.
J. Math. Imaging Vis., 20:89–97, 2004.

[3] P. Charbonnier, L. Blanc-Féraud, G. Aubert, and M. Barlaud.
Deterministic edge-preserving regularization in computed imaging.
IEEE Trans. Image Process., 6:298–311, 1997.

[4] B. Coll, J. Duran, and C. Sbert.
Half-linear regularization for nonconvex image restoration models.
Inverse Probl. Imag., 2014 (in press).

[5] L.I. Rudin, S.J. Osher, and E. Fatemi.
Nonlinear total variation based noise removal algorithms.
Phys. D, 60:259–268, 1992.