Mixed Matrix Norms for Vector Valued Total Variation Regularization

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Vectorial TV Regularization

$$\hat{u} = \arg\min_{u} G_f(u) + \alpha\, TV(u)$$

- Unknown $u : \Omega \to \mathbb{R}^k$
- Data term $G_f(u)$
- Regularization parameter α
- Total variation $TV(u)$

What is the best $TV(u)$ for multichannel images?

The Jacobi matrix at each pixel defines a 3d tensor.
Regularize tensor with mixed matrix norm!

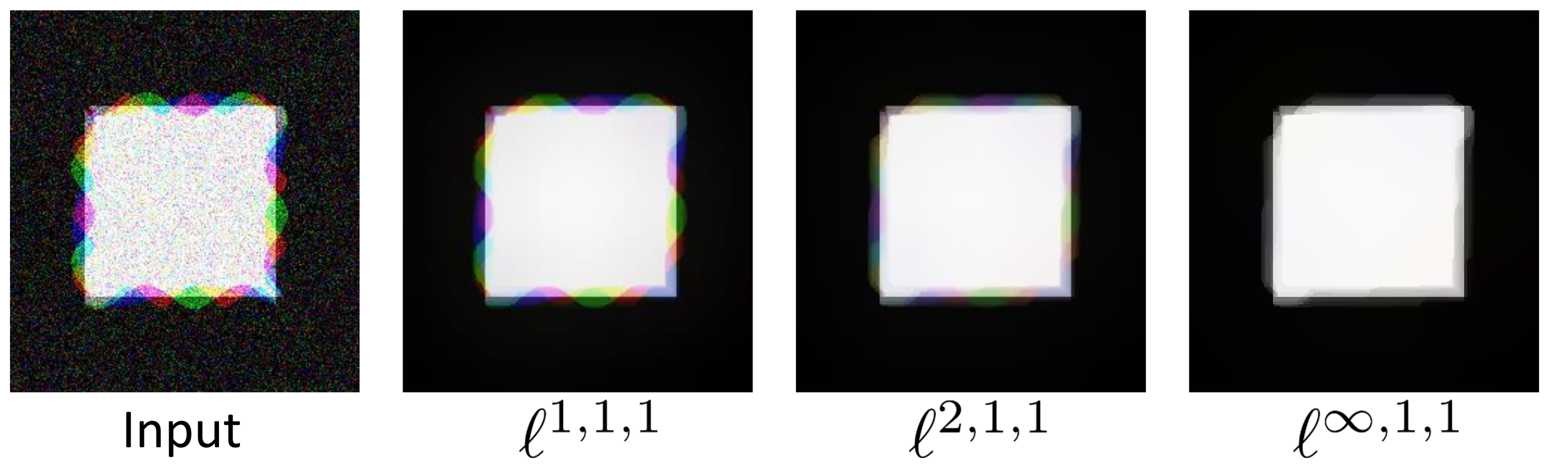
Framework

- Penalize each dimension of *Du* with a different norm!
- For instance, penalize the color dimension with ℓ^p, the resulting 2d matrix with ℓ^q along the derivative dimension, and the resulting vector with ℓ^r

$$(\ell^p(col), \ell^q(der), \ell^r(pix)) = \ell^{p,q,r}(col, der, pix)$$

- With

$$\|A\|_{p,q,r} = \left(\sum_{i=1}^N \left(\sum_{j=1}^M \left(\sum_{k=1}^C |A_{i,j,k}|^p\right)^{q/p}\right)^{r/q}\right)^{1/r}$$



- Of further interest: Schatten-q norms
- The whole framework immediately extends to nonlocal TV (c.f. [1])

Singular Vectors

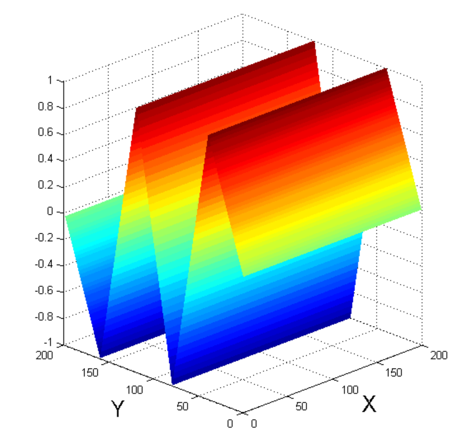
- Singular vectors (introduced in [8]) can be reconstructed exactly
- They reveal the preferred structure of the regularization
- For denoising

$$\lambda u_\lambda \in \partial TV(u_\lambda)$$

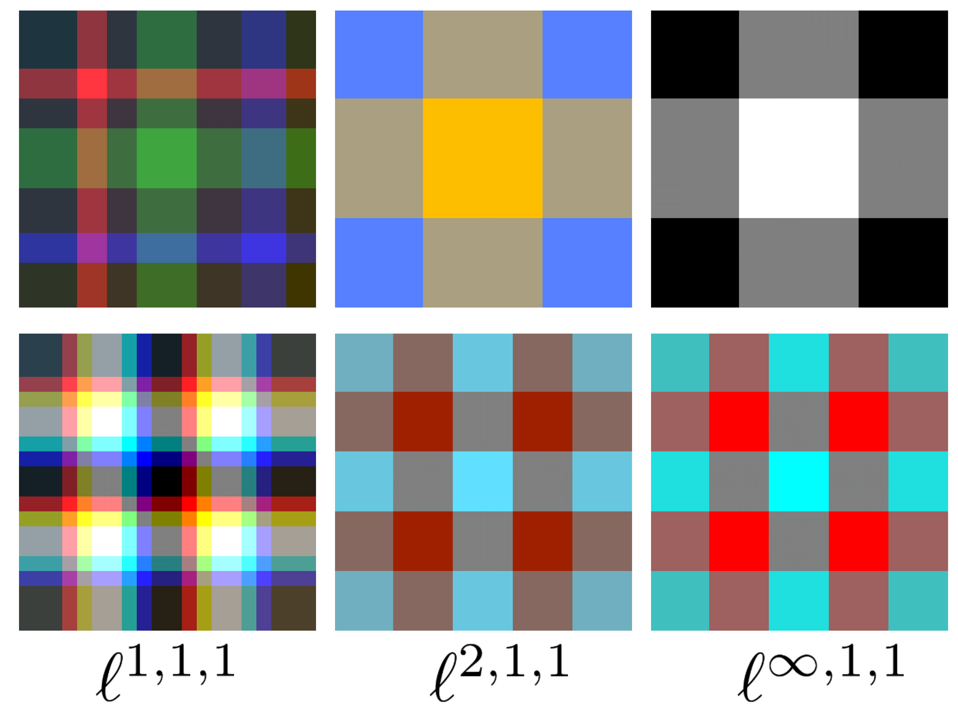
Method	Singular Vectors Channels	Properties
$\|Du\|_{1,1,1}$	$u_k = -c_k^1\, \partial_x l_k^1(x) - c_k^2\, \partial_y l_k^2(y)$	$c_k^r \in \{+1, -1, 0\}$.
$\|Du\|_{2,1,1}$	$u_k = -c_k^1\, \partial_x l^1(x) - c_k^2\, \partial_y l^2(y)$	The piecewise linear functions l^r do not depend on k, $\|c_:^r\|_2 = 1$.
$\|Du\|_{\infty,1,1}$	$u_k = -c_k^1\, \partial_x l^1(x) - c_k^2\, \partial_y l^2(y)$	The piecewise linear functions l^r do not depend on k, $c_k^r \in \{+1, -1, 0\}$.

Function l_k are piecewise linear with the linearity changing at x only if $|l_k(x)| = 1$.

The infinity norm introduces the strongest coupling!



Functions whose divergence generates singular vectors



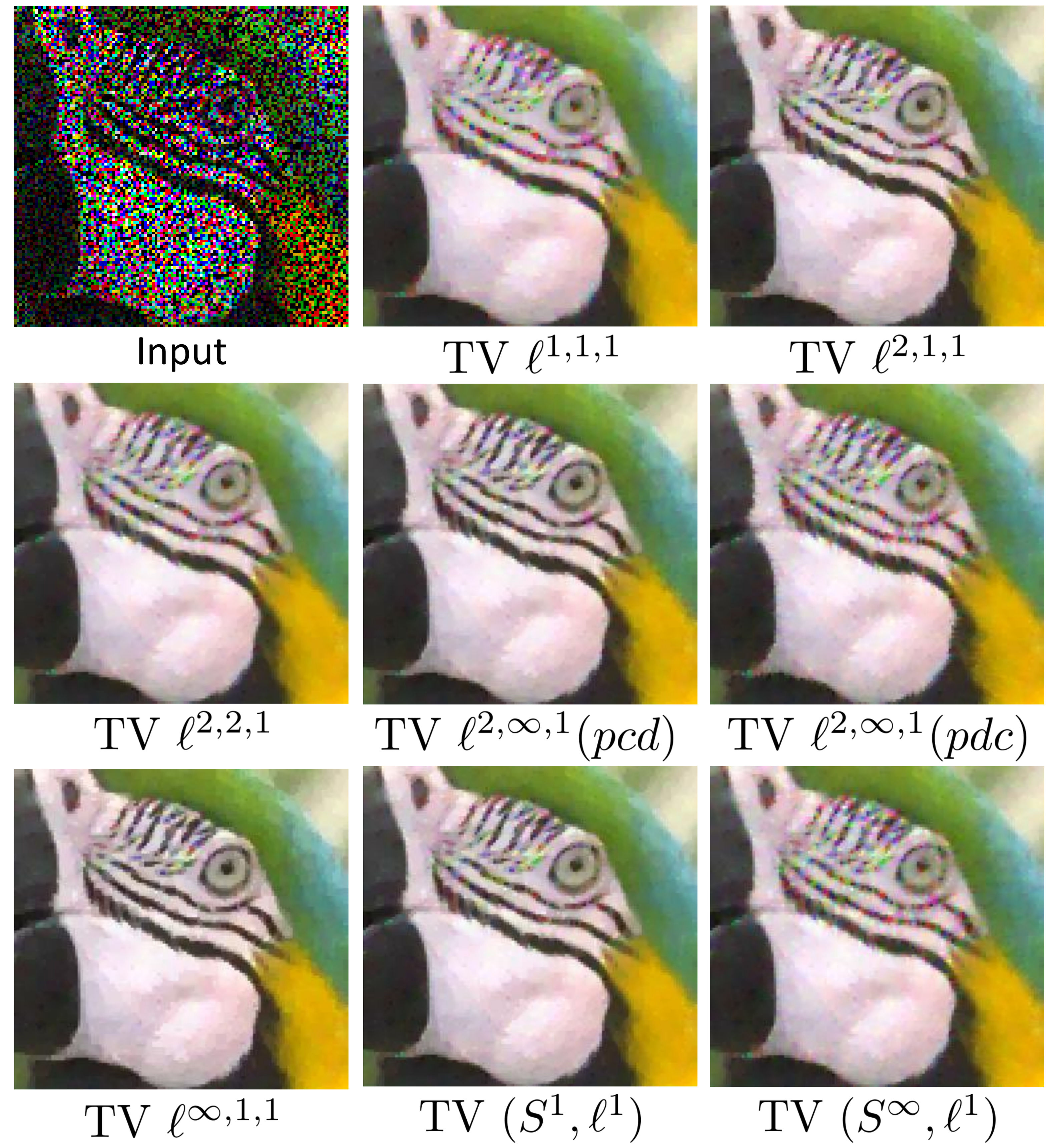
Unification

Background	Formula	Our Notation
Isotropic uncoupled	$\int_\Omega \sum_i \sqrt{(\partial_x u_i)^2 + (\partial_y u_i)^2}\, dx\, dy.$	$\ell^{2,1,1}(der, col, pix)$
Anisotropic uncoupled	$\int_\Omega \sum_i \|\partial_x u_i\| + \|\partial_y u_i\|\, dx\, dy.$	$\ell^{1,1,1}(der, col, pix)$
Proposed in [2]	$\sqrt{\sum_i \left(\int_\Omega \sqrt{(\partial_x u_i)^2 + (\partial_y u_i)^2}\, dx\, dy\right)}$	$\ell^{2,1,2}(der, pix, col)$
Anisotropic version	$\sqrt{\sum_i \left(\int_\Omega \|\partial_x u_i\| + \|\partial_y u_i\|\, dx\, dy\right)}$	$\ell^{1,1,2}(der, pix, col)$
[3,4]	$\int_\Omega \sqrt{\sum_i (\partial_x u_i)^2 + (\partial_y u_i)^2}\, dx\, dy.$	$\ell^{2,2,1}(col, der, pix)$
Anisotropic version 1	$\int_\Omega \sqrt{\sum_i (\|\partial_x u_i\| + \|\partial_y u_i\|)^2}\, dx\, dy.$	$\ell^{1,2,1}(der, col, pix)$
Anisotropic version 2	$\int_\Omega \sqrt{\sum_i (\partial_x u_i)^2} + \sqrt{\sum_i (\partial_y u_i)^2}\, dx\, dy.$	$\ell^{2,1,1}(col, der, pix)$
Related to [5]	$\int_\Omega \max_i \|\partial_x u_i\| + \max_i \|\partial_y u_i\|\, dx\, dy.$	$\ell^{\infty,1,1}(col, der, pix)$
Isotropic version	$\int_\Omega \sqrt{(\max_i \|\partial_x u_i\|)^2 + (\max_i \|\partial_y u_i\|)^2}\, dx\, dy.$	$\ell^{\infty,2,1}(col, der, pix)$
Isotropic variant	$\int_\Omega \max_i \sqrt{\|\partial_x u_i\|)^2 + \|\partial_y u_i\|)^2}\, dx\, dy.$	$\ell^{2,\infty,1}(col, der, pix)$
[4,6]	$\int_\Omega \left\| \begin{pmatrix} (\partial_x u_i)_{i=1,\dots,N} \\ (\partial_y u_i)_{i=1,\dots,N} \end{pmatrix} \right\|_* dx\, dy,$	$(S^1(col, der), \ell^1(pix))$
[4,7]	$\int_\Omega \left\| \begin{pmatrix} (\partial_x u_i)_{i=1,\dots,N} \\ (\partial_y u_i)_{i=1,\dots,N} \end{pmatrix} \right\|_{S^\infty} dx\, dy,$	$(S^\infty(col, der), \ell^1(pix))$

Denoising

Method/Avg. PSNR on	McM (18 img)	Kodak (24 img)	BSDS (20 img)	ARRI (8 img)
$\ell^{1,1,1}(der, col, pix)$	31.15	31.39	31.37	30.31
$\ell^{2,1,1}(der, col, pix)$	31.90	32.32	32.40	31.24
$\ell^{2,2,1}(col, der, pix)$	31.65	32.00	32.21	31.01
$\ell^{\infty,1,1}(col, der, pix)$	31.92	**32.76**	**32.99**	**31.38**
$\ell^{\infty,2,1}(col, der, pix)$	31.72	32.52	32.90	–
$\ell^{2,\infty,1}(der, col, pix)$	31.36	32.19	32.59	31.05
$\ell^{2,\infty,1}(col, der, pix)$	30.92	31.27	31.59	30.34
$(S^1(col, der), \ell^1(pix))$	**32.19**	32.53	32.71	31.30
$(S^\infty(col, der), \ell^1(pix))$	31.01	31.42	31.66	30.46

Inpainting



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