A Projection Algorithm for a Nonconvex Restoration Image Model

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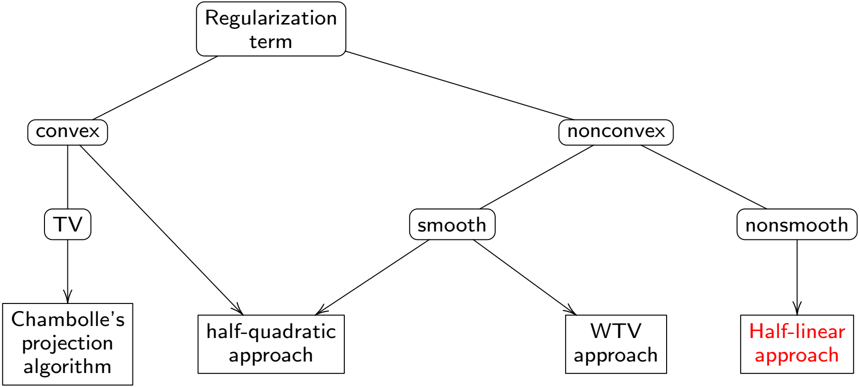
Contents

Inverse problem $\rightarrow$ recover the true image u from an observation of it, u_0.
$\downarrow$
Variational model:

$$\min_{u \in BV(\Omega)} \left\{ \int_\Omega \phi(|\nabla u|) dx + \frac{\lambda}{2} \|u - u_0\|_2^2 \right\},$$

where

- $\phi \rightarrow$ potential function.
- $\int_\Omega \phi(|\nabla u|) dx \rightarrow$ smoothing term.
- $\|u - u_0\|_2^2 \rightarrow$ fidelity to the data.
- $\lambda \rightarrow$ positive weighting constant.


Regularization term
convex
nonconvex
TV
smooth
nonsmooth
Chambolle's projection algorithm
half-quadratic approach
WTV approach
Half-linear approach

Contents

- Discontinuities in images should be preserved.
- **BV-space**:

$$\int_\Omega |\nabla u|\, dx := \sup \left\{ \int_\Omega u(x) \text{div}\,(\xi(x))\, dx \;:\; \xi \in C_c^1(\Omega, \mathbb{R}^2),\; \|\xi\|_\infty \leq 1 \right\}$$

$$BV(\Omega) := \left\{ u \in L^1(\Omega) : \int_\Omega |\nabla u| < \infty \right\}$$

Weighted Total Variation

Let $\omega \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}_+)$. The **weighted total variation (WTV)** of u is given by

$$\int_\Omega \omega(x) |\nabla u| := \sup \left\{ \int_\Omega u \text{ div}\,(\omega \xi) : \xi \in C_c^1(\Omega, \mathbb{R}^N), \|\xi\|_\infty \leq 1 \right\}.$$

- **Discrete linear gradient operator** $\rightarrow (\nabla u)_{ij} = \Big((\nabla u)^1_{ij}, (\nabla u)^2_{ij} \Big)$,

$$(\nabla u)^1_{ij} = \begin{cases} u_{i+1,j} - u_{ij} & \text{if } i < N, \\ 0 & \text{if } i = N, \end{cases} \quad \text{and} \quad (\nabla u)^2_{ij} = \begin{cases} u_{i,j+1} - u_{ij} & \text{if } j < N, \\ 0 & \text{if } j = N. \end{cases}$$

- **Discrete weighted Total Variation**:

$$F(u) = \sum_{1 \leq i,j \leq N} \omega_{ij} \left| (\nabla u)_{ij} \right|.$$

- **Discrete divergence** $\rightarrow \langle -\text{div}\,(\omega p), u \rangle_X = \langle \omega p, \nabla u \rangle_Y \;\; \forall p \in Y, \;\; \forall u \in X$, then:

$$(\text{div}\,(\omega p))_{ij} = \begin{cases} \omega_{ij} p^1_{ij} & \text{if } i = 1, \\ -\omega_{i-1,j} p^1_{i-1,j} & \text{if } i = N, \\ \omega_{ij} p^1_{ij} - \omega_{i-1,j} p^1_{i-1,j} & \text{otherwise,} \end{cases} + \begin{cases} \omega_{ij} p^2_{ij} & \text{if } j = 1, \\ -\omega_{i,j-1} p^2_{i,j-1} & \text{if } j = N, \\ \omega_{ij} p^2_{ij} - \omega_{i,j-1} p^2_{i,j-1} & \text{otherwise.} \end{cases}$$

- **Discrete problem:**

$$\min_{u \in X} F(u) + \frac{\lambda}{2}\|u - u_0\|_2^2. \qquad (1)$$

- The unique solution $u \in X$ of (**??**) is given by[1]

$$\boxed{\mathbf{u} = \mathbf{u_0} - \pi_{\frac{1}{\lambda}\mathbf{K}}(\mathbf{u_0})}$$

where $\pi_{\frac{1}{\lambda}K}(u_0)$ is the projection of u_0 on the set

$$K = \{\mathrm{div}\,(\omega p)\ :\ p \in Y,\ |p_{ij}| \leq 1\ \forall i, j\}.$$

- **Constrained minimization problem:**

$$\min \left\{ \left\| \frac{1}{\lambda}\mathrm{div}\,(\omega p) - u_0 \right\|_2^2\ :\ p \in Y,\ |p_{ij}|^2 - 1 \leq 0\ \forall i, j \right\}.$$

- **Semi-implicit scheme**:

$$p_{i,j}^{k+1} = \frac{p_{i,j}^k + \tau\left(\nabla\left(\mathrm{div}\left(\omega p^k\right) - \lambda u_0\right)\right)_{ij}}{1 + \tau\left|\left(\nabla\left(\mathrm{div}\left(\omega p^k\right) - \lambda u_0\right)\right)_{ij}\right|}.$$

[1] A.Chambolle. An algorithm for total variation minimization and applications. *Journal of Mathematical Imaging and Vision*, 20:89-97,2004.

Contents

Discrete energy:

$$J(u) = \sum_{1 \le i,j \le N} \phi\left(|(\nabla u)_{i,j}|\right) + \frac{\lambda}{2}\|u - u_0\|_2^2, \qquad (2)$$

where ϕ is a positive **nonconvex potential function satisfying $\phi'(0^+) > 0$**.

- (**??**) involves numerical intrincacies:
  - Various local minima.
  - Nonsmooth at the minimizers.
- We propose a **dual formulation of the WTV norm**:
  - Introduce an auxiliary variable ω which determines the location of edges.
  - The *augmented energy* $J^*(u, \omega)$ involves a WTV term respect to u.

1. **WTV approach**[2]

- **Nonconvex but smooth potential functions.**
- Rewrite the regularizing term as a **WTV**:

$$\sum_{1\leq i,j\leq N} \omega_{i,j}^{n+1} |(\nabla u^{n+1})_{i,j}| + \frac{\lambda}{2}\|u^{n+1}-u_0\|_2^2,$$

where

$$\omega^{n+1} := \frac{\phi\left(|\nabla u^n|\right)}{|\nabla u^n|}.$$

- Compute u^{n+1} by **Chambolle's projection algorithm**.
- Numerical intricacies when $|\nabla u| \to 0$ are avoided by $\phi(0) = \phi'(0) = 0$.

Half-linear approach

We rewrite the **nonconvex nonsmooth** regularizing term as a WTV, under assumption $\phi'(0) > 0$.

[2] A.E. Hamidi et al. Weighted and extended total variation for image restoration and decomposition. *Pattern Recognition*, 43(4):960-991, 2005.

2. **Half-quadratic approach**[3].

- Some **convex** and **nonconvex but smooth potential functions**.
- Under the **edge-preservation assumptions**

$$\lim_{t\to+\infty}\frac{\phi'(t)}{t}=0 \quad \text{and} \quad \lim_{t\downarrow 0+}\frac{\phi'(t)}{t}=M,$$

rewrite $\phi(t)=\inf_\omega\left\{\omega t^2+\psi(\omega)\right\}$.

- Semi-implicit scheme to the E-L equation:

$$-\text{div}\left(\frac{\phi'\left(|\nabla u^n|\right)}{|\nabla u^n|}\nabla u^{n+1}\right)+\lambda(u^{n+1}-u_0)=0.$$

- If we set

$$\omega^{n+1}:=\frac{\phi'\left(|\nabla u^n|\right)}{|\nabla u^n|},$$

then the divergence term becomes a **weighted discrete laplacian operator**.

Half-linear approach

Our approach consists in rewriting $\phi(t)=\inf_\omega\left\{\omega t+\psi(\omega)\right\}$:

$$-\text{div}\left(\phi'\left(|\nabla u^n|\right)\frac{\nabla u^{n+1}}{|\nabla u^{n+1}|}\right)+\lambda(u^{n+1}-u_0)=0.$$

By setting $\omega^{n+1}:=\phi'\left(|\nabla u^n|\right)$, the divergence becomes in a **WTV term**.

[3] P. Charbonnier et al. Deterministic edge-preserving regularization in computed imaging. *IEEE Transactions on*, 37(12):2024-2036, 1989

Hypothesis on ϕ:

- **Image analysis conditions**

  A1 ϕ is symmetric on $\mathbb{R}_+$ with $\phi(0) = 0$, $\phi'(0^+) > 0$ and $\phi'(t) > 0$ for each $t \geq 0$.

  A2 $\phi \in \mathcal{C}^2\left(\mathbb{R}_+^*\right)$, with $\mathbb{R}_+^* = (0, +\infty)$.

- **Edge-preserving conditions**

  A3 ϕ' strictly decreases on $\mathbb{R}_+^*$.

  A4 $\lim\limits_{t \to \infty} \phi'(t) = 0$.

  A5 $\lim\limits_{t \downarrow 0^+} \phi'(t) = M$, with $0 < M < +\infty$.

- **Technical hypothesis**

  A6 $\lim\limits_{t \downarrow 0^+} \phi''(t) = \mu$, with $-\infty < \mu \leq 0$.

Theorem

Let ϕ satisfy A1-A5. Then:

i) $\exists\, \psi : (0, M) \to (0, \beta)$ *strictly convex and decreasing, with* $\beta = \lim_{t\to\infty} \phi(t)$, *s.t.*

$$\phi(t) = \inf_{\omega \in (0,M)} \left(\omega t + \psi(\omega)\right).$$

ii) $\forall\, t > 0\ \exists!\, \omega_t \in (0, M)$, *given by* $\omega_t = \phi'(t)$, *s.t.* $\phi(t) = \omega_t t + \psi(\omega_t)$.

- **Augmented energy:**

$$J^*(u, \omega) = \sum_{1\le i,j\le N} \left(\omega_{i,j} |(\nabla u)_{i,j}| + \psi(\omega_{i,j})\right) + \frac{\lambda}{2}\|u - u_0\|_2^2. \qquad (3)$$

- J^* is **convex** in each variable when the other is fixed.
- The unique ω that minimizes (**??**) is

$$\omega = \phi'\left(|\nabla u|\right).$$

Examples of potential functions

$\phi(t)$	ω	M	β	ψ
$\dfrac{t}{1+t}$	$\dfrac{1}{(1+t)^2}$	1	1	$\left(1-\sqrt{\omega}\right)^2$
$\ln(1+t)$	$\dfrac{1}{1+t}$	1	$+\infty$	$\omega - \ln\omega - 1$
$\dfrac{t}{\sqrt{1+t^2}}$	$\dfrac{1}{(1+t^2)^{\frac{3}{2}}}$	1	1	$\left(1-\omega^{\frac{2}{3}}\right)^{\frac{3}{2}}$

- **Half-linear algorithm:**

  $u^0 := u_0$
  repeat
  $\omega^{n+1} = \arg\min_\omega J^*(u^n, \omega)$
  $u^{n+1} = \arg\min_u J^*(u, \omega^{n+1})$
  until convergence
  return $(u^\infty, \omega^\infty)$.

- When u is fixed, the minimum of $J^*(u^n, \omega)$ is

$$\omega^{n+1} = \phi'(|\nabla u^n|).$$

- Once we fix ω, the problem

$$u^{n+1} = \arg\min_u J^*(u, \omega^{n+1})$$

  becomes a **WTV regularization** → Chambolle's projection algorithm.

Theorem

Let ϕ satisfy A1-A6. Then, $\{J^*(u^n, \omega^{n+1})\}_n$ converges and $\|\omega^{n+1} - \omega^n\| \to 0$.

Contents

- Nonconvexity allow us to recover neat edges.
- **Aim** $\rightarrow$ discrete minimizers provide a reconstructed image composed of **constant regions surrounded by sharp edges**.
- People have used

$$\phi_1(t) := \frac{t}{1+t} \quad \text{and} \quad \phi_2(t) := \ln(1+t).$$

- We propose

$$\phi_3(t) := \frac{t}{\sqrt{1+t^2}}.$$

- Nikolova et al.[4] show that, if

$$\phi'' \text{ increases}, \phi''(t) \leq 0, \lim_{t \downarrow 0^+} \phi''(t) < 0 \text{ and } \lim_{t \to \infty} \phi''(t) = 0,$$

then any minizer satisfies

$$\text{either} \quad |(\nabla \hat{u})_{i,j}| = 0 \quad \text{or} \quad |(\nabla \hat{u})_{i,j}| \geq \theta, \quad \forall\, 1 \leq i,j \leq N.$$

- ϕ_1 and ϕ_2 satisfy the above requirement but ϕ_3 does not.

Theorem

Let $\hat{u}$ be any minimizer of the discrete functional with ϕ_3. Then,

$$\textit{either} \quad |(\nabla \hat{u})_{i,j}| = 0 \quad \textit{or} \quad |(\nabla \hat{u})_{i,j}| \geq \theta, \quad \forall\, 1 \leq i,j \leq N,$$

where $\theta > 0$ is the unique real value that solves

$$\xi(\theta) = -\lambda \kappa^{-4} \|\hat{u}\|_2^2,$$

where $\xi(t) := \frac{\phi_3''(t)}{t^2}$ and $\kappa := \min\{|(\nabla \hat{u})_{i,j}| > 0\}$.

[4] Nikolova et al. Fast nonconvex nonsmooth minimization methods for image restoration and reconstruction. *Image Processing, IEEE Transactions on*, 19(12):3073-3088, 2010.

Contents

- Small gradients involve **extremely small weights**.
- Introduce a **thresholding parameter** α for the intensity gradient, $\frac{|\nabla u|}{\alpha}$, in order to **penalize small oscillations**.

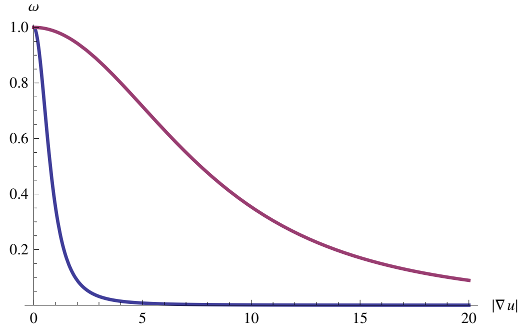


Figure : Graphics of the weighting function associated to ϕ_3 without scaling and after scaling.

- **Chambolle's iterations** stop when

$$\max_{i,j} |p^k_{i,j} - p^{k+1}_{i,j}| < 10^{-2}.$$

- **Half-linear algorithm** proceed until

$$\frac{\|u^{n+1} - u^n\|_2^2}{\|u^n\|_2^2} < 10^{-20}.$$

- **Parameters**: $\omega^0 = \phi'(|\nabla u_0|)$, $\alpha = 20$ and $\lambda = 0.1$.
- We plot ω in gray scale in order to show that it acts as an **edge detector**.
- We give the **topographic maps** of the segmented images.

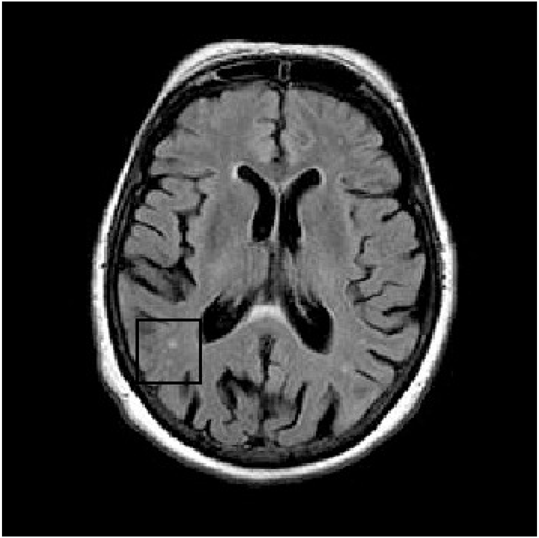


Figure : Original brain image.

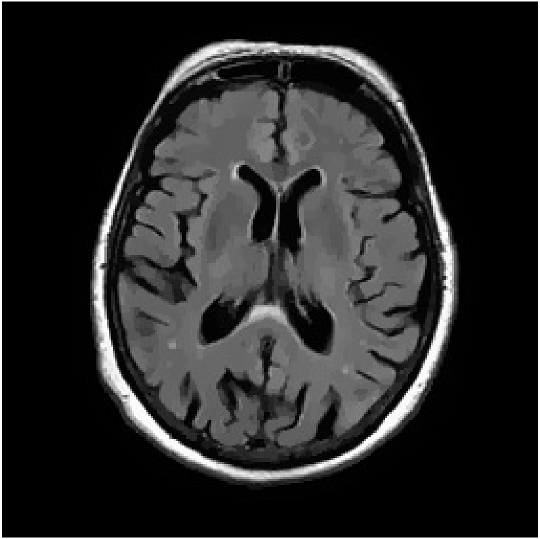


Figure : Segmented brain image with ϕ_1 as potential function.

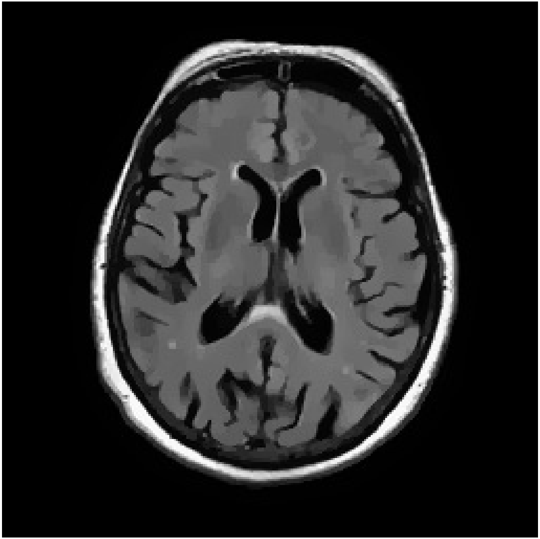


Figure : Segmented brain image with ϕ_2 as potential function.

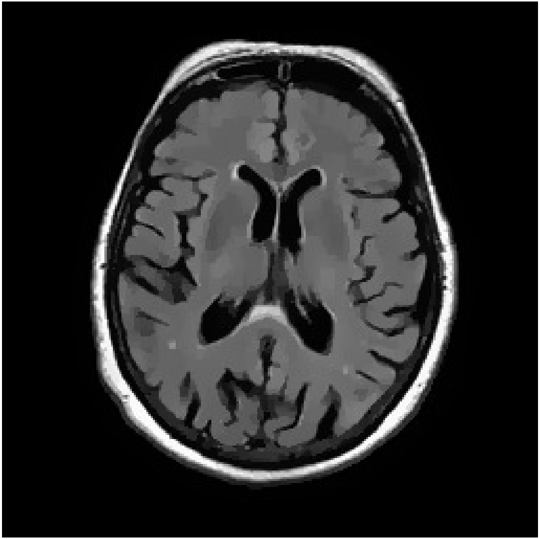


Figure : Segmented brain image with ϕ_3 as potential function.

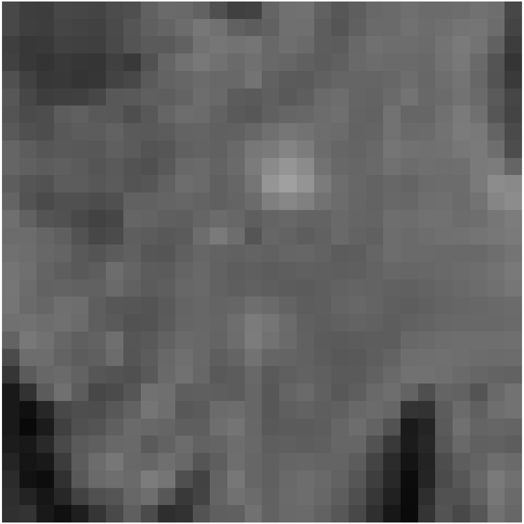


Figure : Zoom in on the selected detail in the brain image.

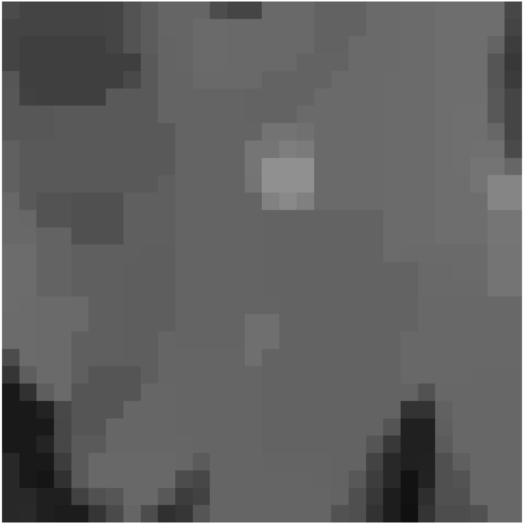


Figure : Segmented brain image with ϕ_1 as potential function.

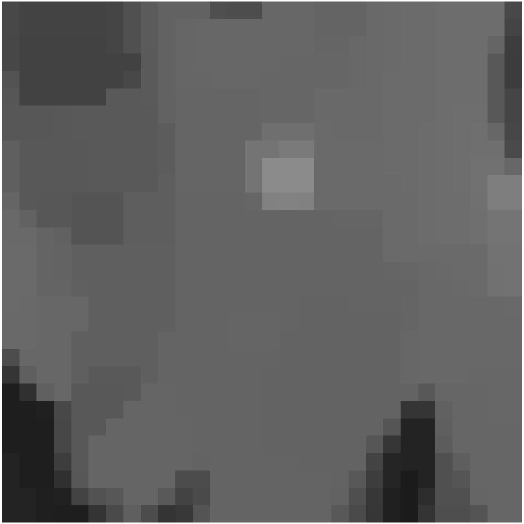


Figure : Segmented brain image with ϕ_2 as potential function.

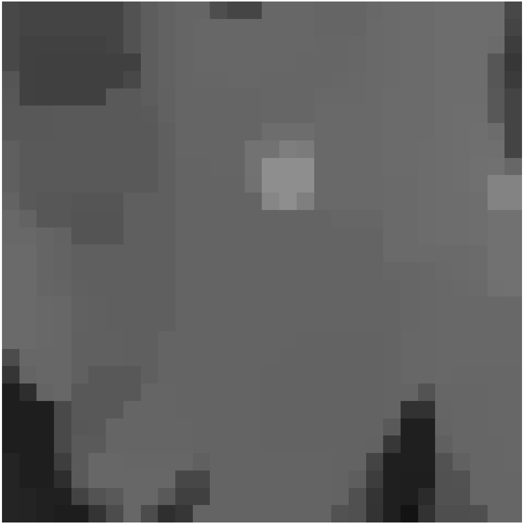


Figure : Segmented brain image with ϕ_3 as potential function.

Figure : Original house image.

Figure : Segmented house image with ϕ_3 as potential function.

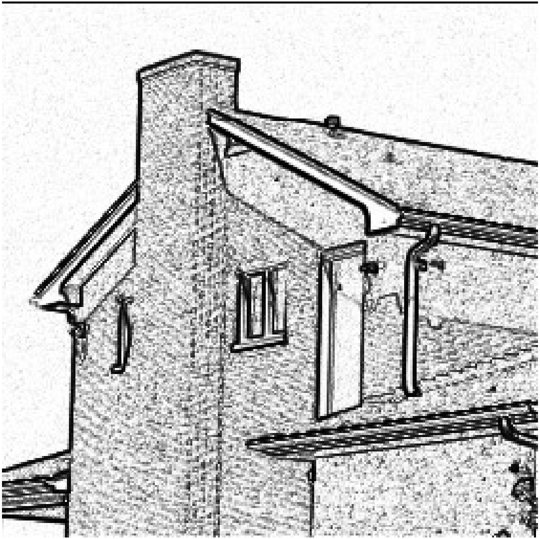


Figure : Value of initial weight for house image.

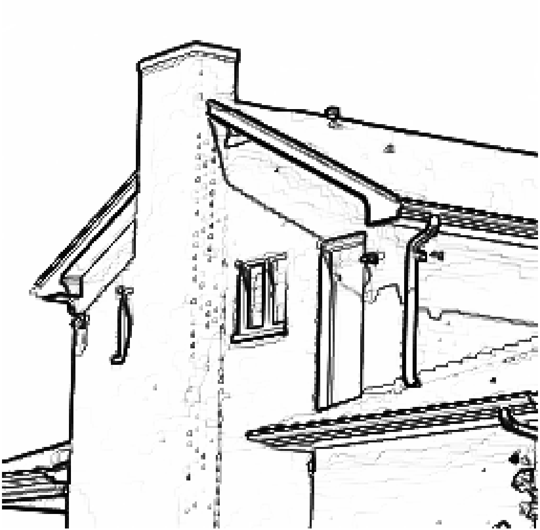


Figure : Value of weight after step 1 for house image.

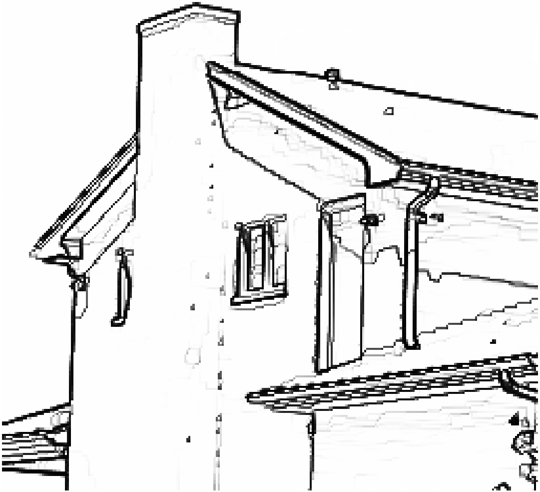


Figure : Value of weight after step 5 for house image.

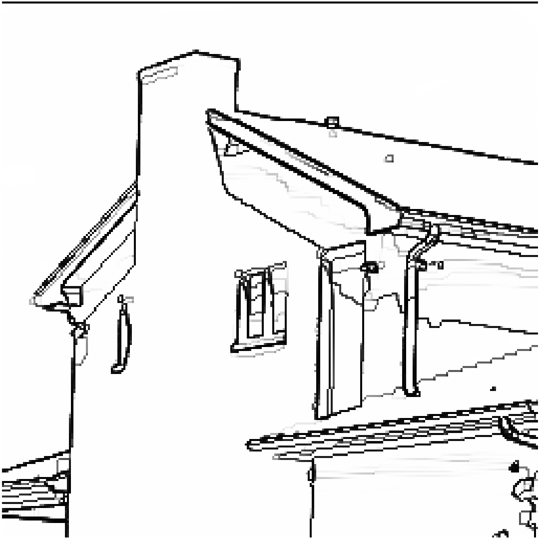


Figure : Value of the final weight after applying half-linear algorithm for house image.

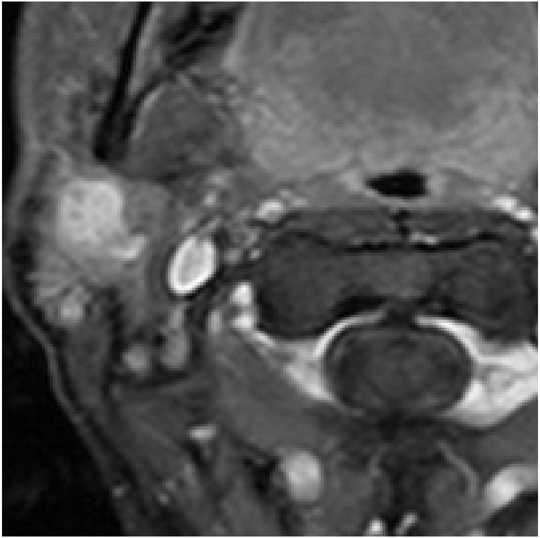


Figure : Original MRI image.

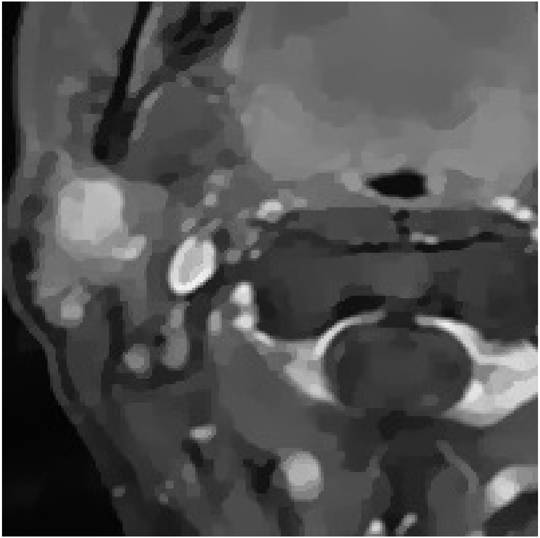


Figure : Segmented MRI image with ϕ_3 as potential function.

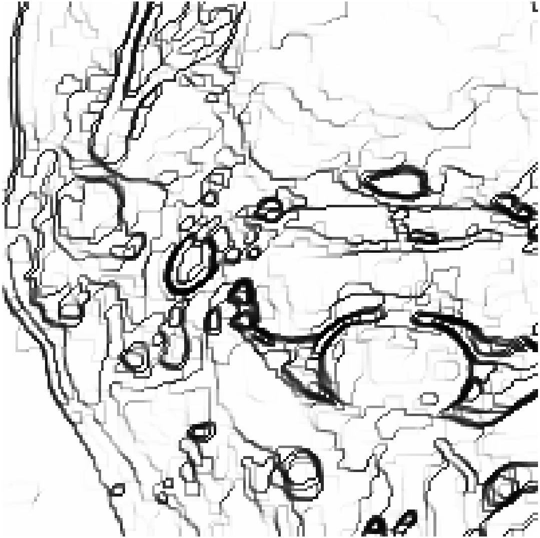


Figure : Value of the final weight for MRI image.

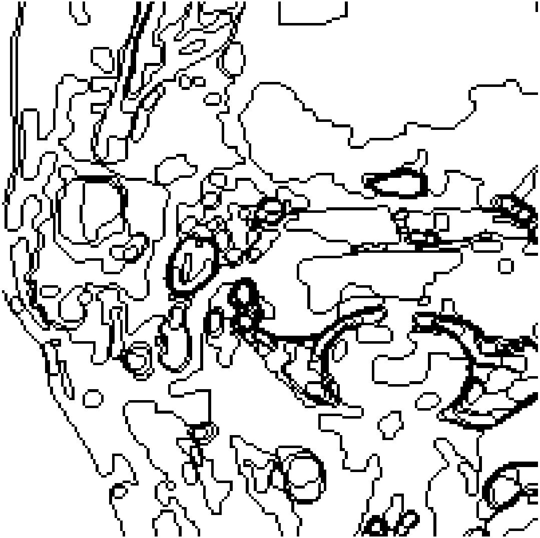


Figure : Topographic map of the segmented MRI image.

Figure : Original house image.

Figure : Segmented house image with ϕ_3 as potential function.

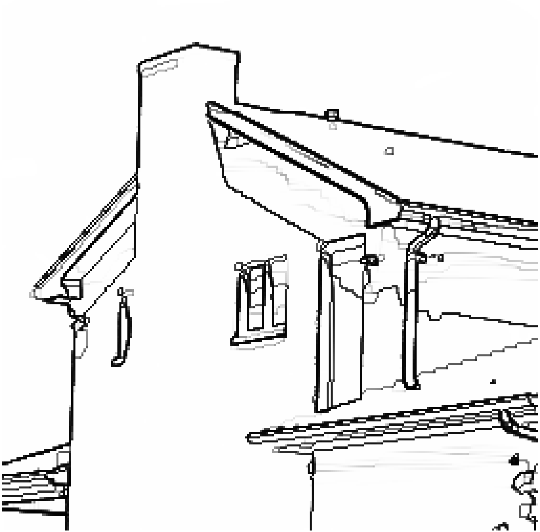


Figure : Value of the final weight for house image.

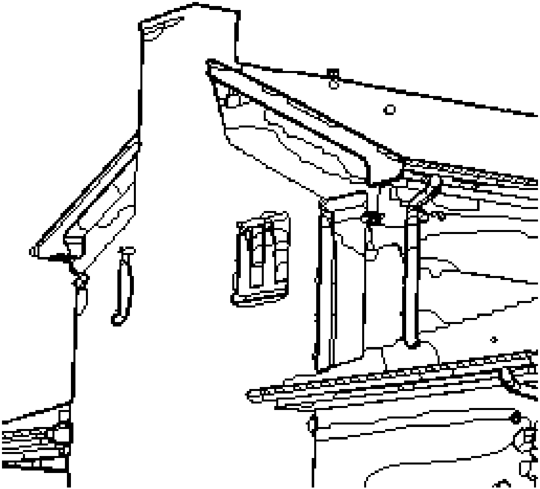


Figure : Topographic map of the segmented house image.

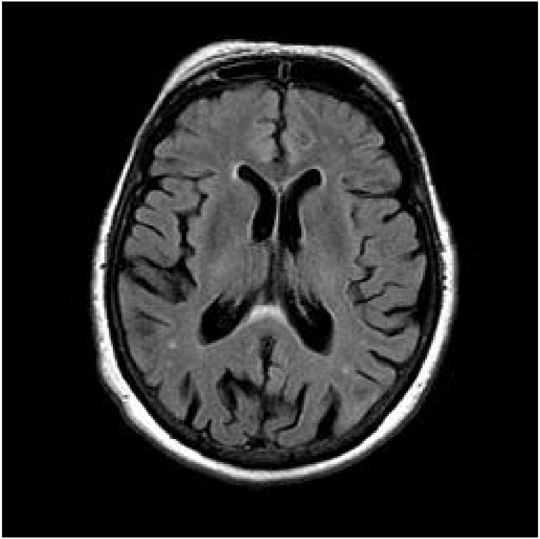


Figure : Original brain image.

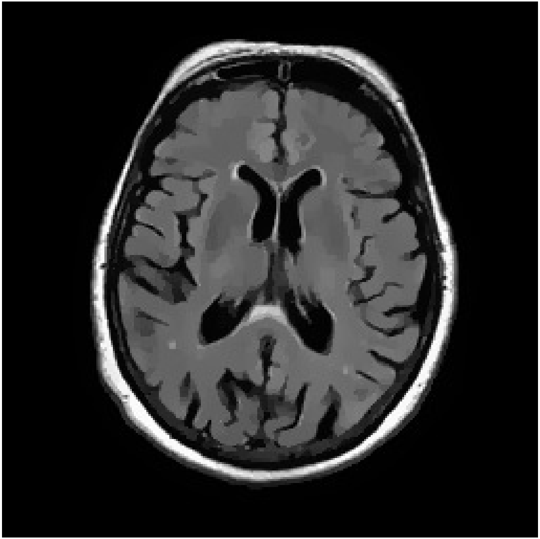


Figure : Segmented brain image with ϕ_3 as potential function.

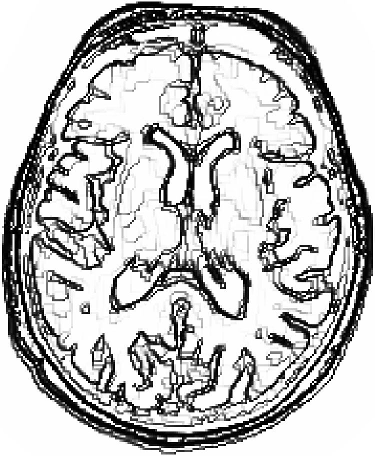


Figure : Value of the final weight for brain image.

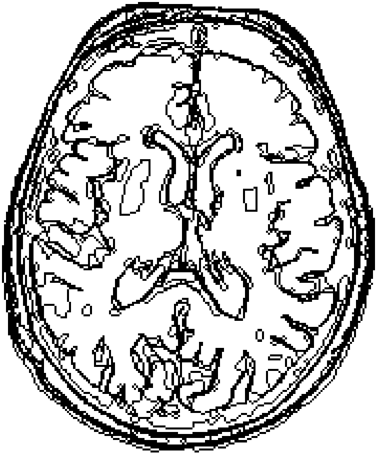


Figure : Topographic map of the segmented brain image.

- Algorithms for **TV-denoising** $\rightarrow$ **Chambolle's projection algorithm**.
- We only based on the visual quality of the restored images to judge the performance of the method.
- We have added noise of standard deviation $\sigma = 10$ to an original image.
- **Parameters:** $\alpha = 50$ and $\lambda = 0.1$.

Figure : Original Lena image.

Figure : Noisy Lena image.

Figure : Reconstructed Lena image after applying half-linear algorithm with ϕ_3.

Figure : Reconstructed Lena image after applying Chambolle's algorithm for TV-denoising.

Contents

Conclusions

- Nonconvex nonsmooth image models allows one to better recover the discontinuities.
- We have proved that the discrete minimizer of nonconvex and nonsmooth functionals provides a reconstructed image composed of constant regions surrounded by sharp edges.
- Numerically, characterizing the solution of these minimization problems is not an easy task.
- We have introduced a dual variable which determines the location of edges and it allows to rewrite the functional as a WTV term.
- We have conceived a half-linear algorithm based on alternate minimization on u and ω. We have obtainded the following:
  - This method allow us to recover the segmentation from the auxiliary variable.
  - The final solution gives us a more or less piecewise constant image.
  - In the case of noisy images, our results are comparable to the TV-denoising.

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