Collaborative Regularization Approaches in Multi-Channel Variational Imaging

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Joint work with

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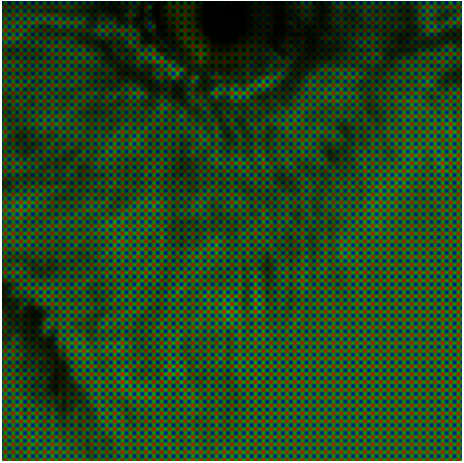
Universitat de les Illes Balears

Ill-Posed Inverse Problems in Image Processing

What do you see?



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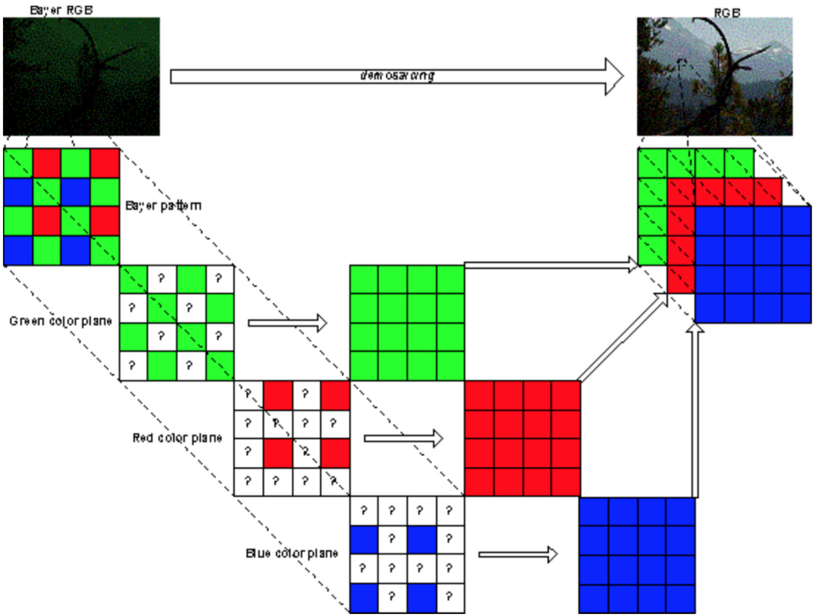
Bayer RGB
demosaicing
RGB
Bayer pattern
Green color plane
Red color plane
Blue color plane

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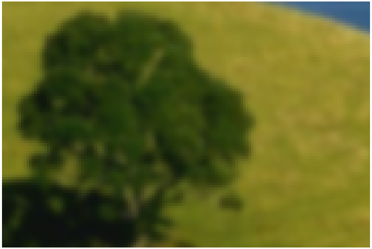

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- Regularization methods handles ill-posedness by introducing prior knowledge on u, usually assuming smooth solutions.
- In the variational framework the regularized solution is computed as

$$\hat{u} = \arg\min_u R(u) + \lambda G_f(u),$$

where $R(u)$ is the regularization term, $G_f(u)$ is the data-fidelity term and $\lambda \geq 0$ is a trade-off parameter.

Regularization Methods
Total Variation

- Consider the inverse problem

$$\min_{u\in \mathrm{BV}(\Omega,\boldsymbol{R})} R(u) + \frac{\lambda}{2}\,\|Au - f\|_2^2\,,$$

with $\Omega \subset \boldsymbol{R}^M$, $f \in L^2\,(\Omega, \boldsymbol{R})$ and a linear operator $A : L^2(\Omega) \to L^2(\Omega)$.

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- A popular regularizer is the total variation [Rudin, Osher, Fatemi '92]:

$$R(u) = \mathrm{TV}(u) = \underbrace{\int_\Omega \|\nabla u(x)\|_2 \, dx}_{u \in \mathcal{C}^1(\Omega,R)} = \underbrace{\sup_{\boldsymbol{\xi} \in \Xi} \left\{ \int_\Omega u \operatorname{div} \boldsymbol{\xi} \, dx \right\}}_{u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega,R)},$$

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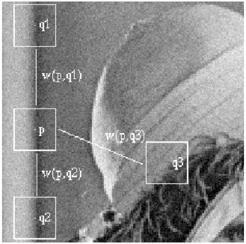
- Anisotropic TV follows from using the L^1 norm on the dual variable $\boldsymbol{\xi}$.
- TV regularizes the image without smoothing the boundaries of the objects, but fails to recover fine structures and texture.

Regularization Methods

Nonlocal techniques

- Nonlocal means denoising algorithm [Buades, Coll, Morel '05]:



$$NL[u](x) = \frac{1}{\int_\Omega \omega_f(x,y)\,dy} \int_\Omega \omega_f(x,y)u(y)\,dy$$

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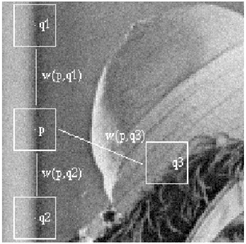

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$$\omega_f(x,y) = e^{-\frac{d_\rho(f(x),f(y))}{h^2}},$$

with patch-based distance:

$$d_\rho(f(x), f(y)) = \int_\Omega G_\rho(t) |f(x+t) - f(y+t)|^2 dt.$$

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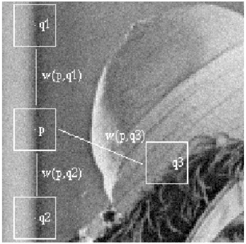

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- Regularity assumption: natural images are self-similar.

Real image demosaicking

Real image denoising

Video denoising

- Neighborhood filters as nonlocal regularization [Gilboa, Osher '08]:

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- Nonlocal operators with non-symmetric weights [D., Buades, Coll, Sbert '14]:

$$\nabla_\omega u(x, y) = (u(y) - u(x)) \sqrt{\omega_f(x, y)},$$
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How can we generalize TV and NLTV to vector-valued images?

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Classical approaches

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- Global channel coupling [Sapiro, Ringach '96]:

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- Spectral norm coupling [Goldluecke, Strekalovskiy, Cremers '12]:

$$\mathrm{VTV}(\mathbf{u}) = \int_\Omega \|\nabla \mathbf{u}\|_{\sigma_1} \, dx = \sup_{(\boldsymbol{\xi},\boldsymbol{\eta})\in\Xi} \left\{ \sum_{k=1}^{C} \int_\Omega u_k \,\mathrm{div}\,(\boldsymbol{\eta}_k \boldsymbol{\xi}) \, dx \right\},$$

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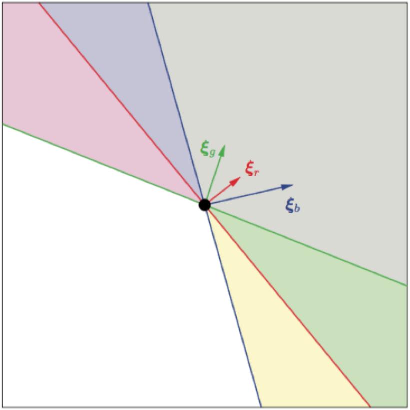
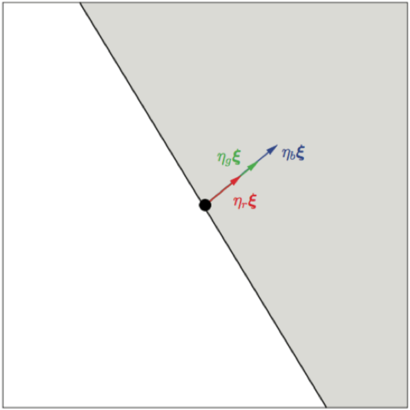


$$\int_\Omega \|\nabla \mathbf{u}\|_F \, dx$$

Different edge direction
Channel-by-channel weights

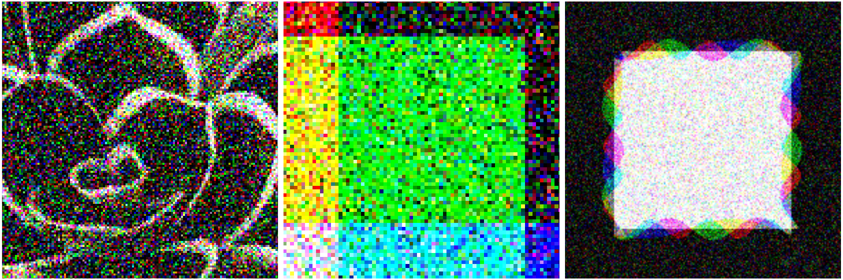


$$\int_\Omega \|\nabla \mathbf{u}\|_{\sigma_1} \, dx$$

Common edge direction
Channel-by-channel weights

Vectorial Total Variation

Which is the best VTV for vector-valued images?



Vectorial Total Variation
- coupling spatial derivatives
  - isotropic diffusion
  - anisotropic diffusion
- coupling color channels
  - ℓ^p – type coupling
  - Spectral coupling

Collaborative Total Variation for Multi-Channel Images

Proposed framework

- Represent an image $\mathbf{u}$ with N pixels and C spectral channels by the matrix

$$\mathbf{u} = (u_1, \dots, u_C) \in \boldsymbol{R}^{N \times C} \;\text{ s.t. }\; u_k \in \boldsymbol{R}^N, \forall k \in \{1, \dots, C\}.$$

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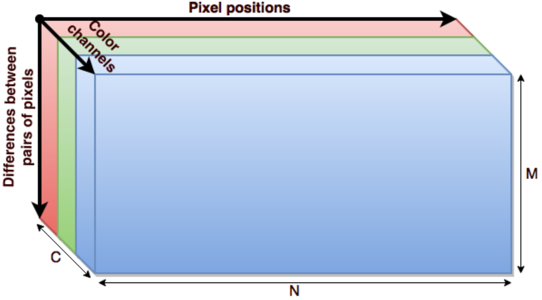
- The Jacobi matrix at each pixel defines a 3D tensor given by

$$D\mathbf{u} \equiv (Du)_{i,j,k} \in \boldsymbol{R}^{N \times M \times C},$$

with M directional derivatives.



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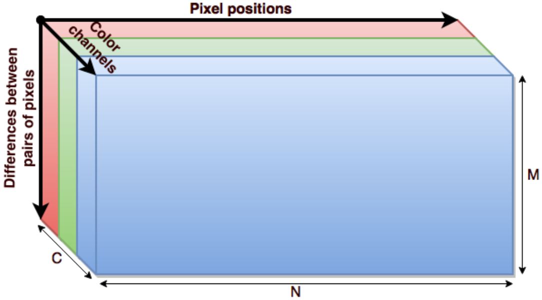

- Regularize $D\mathbf{u}$ by penalizing each dimension with a different norm.

Example (Local gradient operator)

Consider a color image $\mathbf{u} \in \mathbb{R}^{N\times 3}$ *and the local gradient computed at each pixel via forward differences. Then, the submatrix obtained by fixing the n−th pixel is*

$$\begin{pmatrix} u_{n+1,1} - u_{n,1} & u_{n+1,2} - u_{n,2} & u_{n+1,3} - u_{n,3} \\ u_{n+N_w,1} - u_{n,1} & u_{n+N_w,2} - u_{n,2} & u_{n+N_w,3} - u_{n,3} \end{pmatrix}$$

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Example (Nonlocal gradient operator)

Consider a color image $\mathbf{u} \in \mathbb{R}^{4\times 3}$ *and compute the nonlocal gradient. If we fix the* $k-$*th channel, the submatrix along pixel and derivative dimensions is*

$$\begin{pmatrix} 0 & \omega_{1,2}\left(u_{1,k} - u_{2,k}\right) & \omega_{1,3}\left(u_{1,k} - u_{3,k}\right) & \omega_{1,4}\left(u_{1,k} - u_{4,k}\right) \\ \omega_{2,1}\left(u_{2,k} - u_{1,k}\right) & 0 & \omega_{2,3}\left(u_{2,k} - u_{3,k}\right) & \omega_{2,4}\left(u_{2,k} - u_{4,k}\right) \\ \omega_{3,1}\left(u_{3,k} - u_{1,k}\right) & \omega_{3,2}\left(u_{3,k} - u_{2,k}\right) & 0 & \omega_{3,4}\left(u_{3,k} - u_{4,k}\right) \\ \omega_{4,1}\left(u_{4,k} - u_{1,k}\right) & \omega_{4,2}\left(u_{4,k} - u_{2,k}\right) & \omega_{4,3}\left(u_{4,k} - u_{3,k}\right) & 0 \end{pmatrix}$$

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- In general, the nonlocal 3D tensor is of size $N \times N \times C$. However, ω is usually sparse.

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Example (Nonlocal gradient operator)

Consider a color image $\mathbf{u} \in \mathbb{R}^{4\times 3}$ *and compute the nonlocal gradient. If we fix the* $k-$*th channel, the submatrix along pixel and derivative dimensions is*

$$\begin{pmatrix} 0 & \omega_{1,2}\left(u_{1,k} - u_{2,k}\right) & \omega_{1,3}\left(u_{1,k} - u_{3,k}\right) & \omega_{1,4}\left(u_{1,k} - u_{4,k}\right) \\ \omega_{2,1}\left(u_{2,k} - u_{1,k}\right) & 0 & \omega_{2,3}\left(u_{2,k} - u_{3,k}\right) & \omega_{2,4}\left(u_{2,k} - u_{4,k}\right) \\ \omega_{3,1}\left(u_{3,k} - u_{1,k}\right) & \omega_{3,2}\left(u_{3,k} - u_{2,k}\right) & 0 & \omega_{3,4}\left(u_{3,k} - u_{4,k}\right) \\ \omega_{4,1}\left(u_{4,k} - u_{1,k}\right) & \omega_{4,2}\left(u_{4,k} - u_{2,k}\right) & \omega_{4,3}\left(u_{4,k} - u_{3,k}\right) & 0 \end{pmatrix}$$

- In general, the nonlocal 3D tensor is of size $N \times N \times C$. However, ω is usually sparse.
- Local gradient is a particular case of the nonlocal gradient by taking

$$\omega_{i,j} = \begin{cases} 1 & \text{if } j \text{ is the right or lower neighbour of } i, \\ 0 & \text{otherwise.} \end{cases}$$

Collaborative Total Variation for Multi-Channel Images

Collaborative sparsity enforcing norms

- The Jacobi matrix at each pixel defines a 3D tensor which can be regularized by penalizing each of its dimensions with a different norm.

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Definition

Let $\|\cdot\|_a : \boldsymbol{R}^N \to \boldsymbol{R}$ be any vector norm and $\|\cdot\|_{\vec{b}} : \boldsymbol{R}^{M\times C} \to \boldsymbol{R}$ any matrix norm. Then, the collaborative norm of $A \in \boldsymbol{R}^{N\times M\times C}$ is defined as

$$\|A\|_{\vec{b},a} = \|v\|_a, \quad \text{with} \quad v_i = \|A_{i,:,:}\|_{\vec{b}}, \ \forall i \in \{1,\dots,N\},$$

where $A_{i,:,:}$ is the submatrix obtained by stacking the second and third dimensions of A at ith position.

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Example ($\ell^{p,q,r}$ norms)

Let $A \in \boldsymbol{R}^{N\times M\times C}$ and consider $\|\cdot\|_{\vec{b}} = \ell^{p,q}$ and $\|\cdot\|_a = \ell^r$. Then, the $\ell^{p,q,r}$ norm is

$$\|A\|_{p,q,r} = \left(\sum_{i=1}^{N} \left(\sum_{j=1}^{M} \left(\sum_{k=1}^{C} |A_{i,j,k}|^p \right)^{q/p} \right)^{r/q} \right)^{1/r}$$

Example ((S^p, ℓ^q) norm)

Let $A \in \mathbf{R}^{N\times M\times C}$ and consider $\|\cdot\|_{\vec{b}} = S^p$ and $\|\cdot\|_a = \ell^q$. Then the (S^p, ℓ^q) norm is

$$(S^p, \ell^q)(A) = \left(\sum_{i=1}^{N} \left\| \begin{pmatrix} A_{i,1,1} & \cdots & A_{i,1,C} \\ \vdots & \ddots & \vdots \\ A_{i,M,1} & \cdots & A_{i,M,C} \end{pmatrix} \right\|_{S^p}^{q} \right)^{1/q}$$

- Schatten $p-$norms:
    - Fix a pixel location and consider the submatrix obtained by looking at the channel and derivative dimensions.
    - Compute SVD and penalize the singular values with an ℓ^p-norm:
        - $p = 1 \rightarrow$ nuclear norm, a convex relaxation of rank minimization.
        - $p = 2 \rightarrow$ Frobenius norm.
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- Any transform along each of the dimensions, in particular, color space transforms, can be applied before CTV.

Collaborative Total Variation for Multi-Channel Images
A unified framework for Vectorial Total Variation

Continuous Formulation	Our Framework
$\int_\Omega \sum_{k=1}^{C} \sqrt{(\partial_{x_1} u_k(x))^2 + (\partial_{x_2} u_k(x))^2} dx$	$\ell^{2,1,1}(der, col, pix)$
$\int_\Omega \sum_{k=1}^{C} \left(\lvert\partial_{x_1} u_k(x)\rvert + \lvert\partial_{x_2} u_k(x)\rvert\right) dx$	$\ell^{1,1,1}(der, col, pix)$
$\sqrt{\sum_{k=1}^{C} \left(\int_\Omega \sqrt{(\partial_{x_1} u_k(x))^2 + (\partial_{x_2} u_k(x))^2} dx\right)^2}$	$\ell^{2,1,2}(der, pix, col)$
$\sqrt{\sum_{k=1}^{C} \left(\int_\Omega \left(\lvert\partial_{x_1} u_k(x)\rvert + \lvert\partial_{x_2} u_k(x)\rvert\right) dx\right)^2}$	$\ell^{1,1,2}(der, pix, col)$
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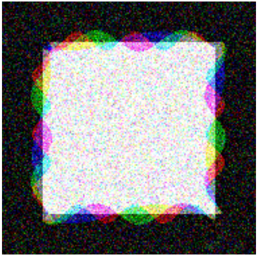
$\int_\Omega \sqrt{\sum_{k=1}^{C} \left(\lvert\partial_{x_1} u_k(x)\rvert + \lvert\partial_{x_2} u_k(x)\rvert\right)^2} dx$	$\ell^{1,2,1}(der, col, pix)$
$\int_\Omega \left(\sqrt{\sum_{k=1}^{C} \left(\partial_{x_1} u_k(x)\right)^2} + \sqrt{\sum_{k=1}^{C} \left(\partial_{x_2} u_k(x)\right)^2}\right) dx$	$\ell^{2,1,1}(col, der, pix)$
$\int_\Omega \left(\max_{1\leq k\leq C} \lvert\partial_{x_1} u_k(x)\rvert + \max_{1\leq k\leq C} \lvert\partial_{x_2} u_k(x)\rvert\right) dx$	$\ell^{\infty,1,1}(col, der, pix)$

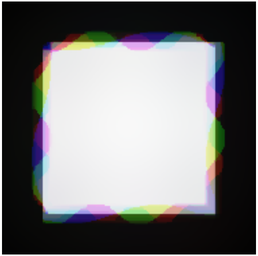
Continuous Formulation	Our Framework
$\int_\Omega \sqrt{\big(\max_{1\le k\le C} \lvert\partial_{x_1} u_k(x)\rvert\big)^2 + \big(\max_{1\le k\le C} \lvert\partial_{x_2} u_k(x)\rvert\big)^2}\, dx$	$\ell^{\infty,2,1}(col, der, pix)$
$\int_\Omega \max_{1\le k\le C} \sqrt{\big(\partial_{x_1} u_k(x)\big)^2 + \big(\partial_{x_2} u_k(x)\big)^2}\, dx$	$\ell^{2,\infty,1}(der, col, pix)$
$\int_\Omega \max\left\{\max_{1\le k\le C} \lvert\partial_{x_1} u_k(x)\rvert, \max_{1\le k\le C} \lvert\partial_{x_2} u_k(x)\rvert\right\} dx$	$\ell^{\infty,\infty,1}(col, der, pix)$
$\int_\Omega \left(\sqrt{\lambda^+(x)} + \sqrt{\lambda^-(x)}\right) dx$	$(S^1(col, der), \ell^1(pix))$
$\int_\Omega \sqrt{\lambda^+(x)}\, dx$	$(S^\infty(col, der), \ell^1(pix))$
$\int_\Omega \left(\sum_{k=1}^{C} \sqrt{\int_\Omega (u_k(y) - u_k(x))^2\, \omega(x,y)\, dy}\right) dx$	$\ell^{2,1,1}_\omega(der, col, pix)$
$\int_\Omega \left(\sum_{k=1}^{C} \int_\Omega \lvert u(y) - u(x)\rvert \sqrt{\omega(x,y)}\, dy\right) dx$	$\ell^{1,1,1}_\omega(der, col, pix)$
$\sqrt{\sum_{k=1}^{C} \left(\int_\Omega \sqrt{\int_\Omega (u_k(y) - u_k(x))^2\, \omega(x,y)\, dy}\, dx\right)^2}$	$\ell^{2,1,2}_\omega(der, pix, col)$
$\int_\Omega \int_\Omega \sqrt{\sum_{k=1}^{C} (u_k(y) - u_k(x))^2\, \omega(x,y)}\, dy\, dx$	$\ell^{2,1,1}_\omega(col, der, pix)$
$\int_\Omega \sqrt{\int_\Omega \sum_{k=1}^{C} (u_k(y) - u_k(x))^2\, \omega(x,y)\, dy}\, dx$	$\ell^{2,2,1}_\omega(col, der, pix)$
$\int_\Omega \int_\Omega \max_{1\le k\le C} \Big((u_k(y) - u_k(x))^2\, \omega(x,y)\Big)\, dy\, dx$	$\ell^{\infty,1,1}_\omega(col, der, pix)$

Which is the Best Channel Coupling?

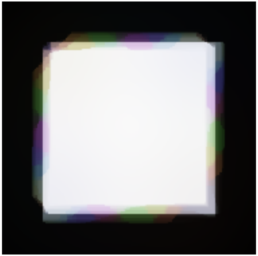
Inter-channel correlation



Noisy

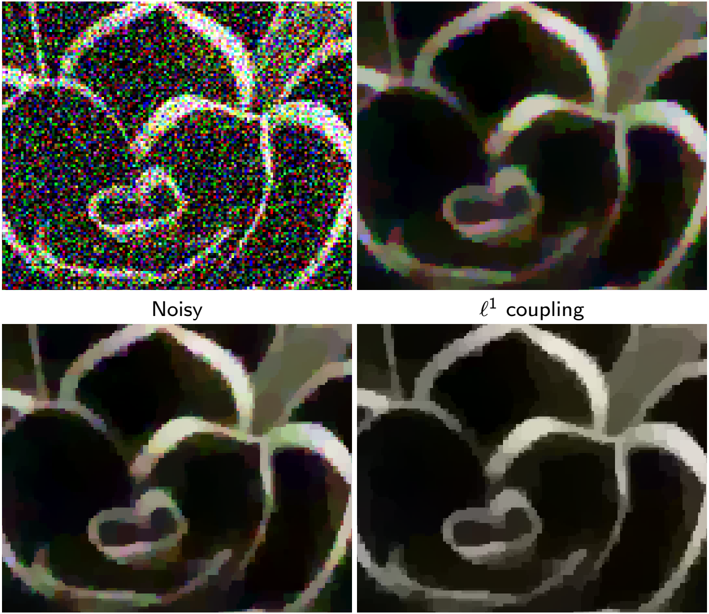


ℓ^1 coupling



ℓ^2 coupling

ℓ^∞ coupling



Noisy

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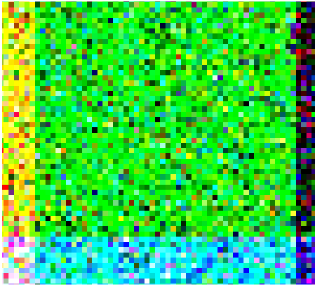
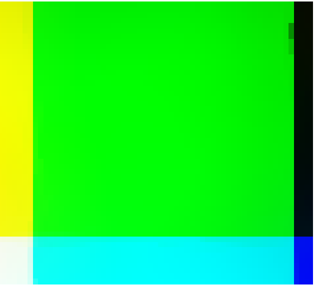


Noisy



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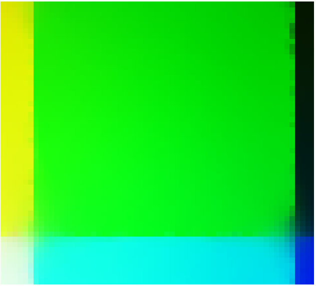

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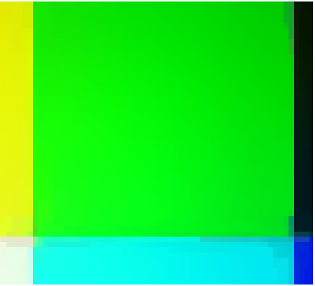

ℓ^∞ coupling

Which is the Best Channel Coupling?

Singular vector analysis

Definition

Let F be a convex regularization s.t. $\partial F(\mathbf{u}) \neq \emptyset$ at any $\mathbf{u} \in \operatorname{dom} F$. Then, every function $\mathbf{u}_\lambda$ s.t. $\|\mathbf{u}_\lambda\| = 1$ and $\lambda \mathbf{u}_\lambda \in \partial F(\mathbf{u}_\lambda)$ is called a singular vector of F with singular value λ.

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- A signal can be restored well if it is a singular vector of F [Benning, Burger '13].

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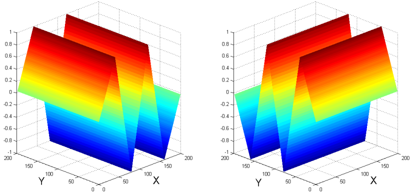
- A signal can be restored well if it is a singular vector of F [Benning, Burger '13].
- Singular vectors of CTV:

$$\mathbf{u} \in \partial \|D\mathbf{u}\|_{\vec{b},a} \Leftrightarrow \mathbf{u} = D^\top \mathbf{z}, \text{ with } \mathbf{z} \in \partial_{D\mathbf{u}}\left(\|D\mathbf{u}\|_{\vec{b},a}\right).$$

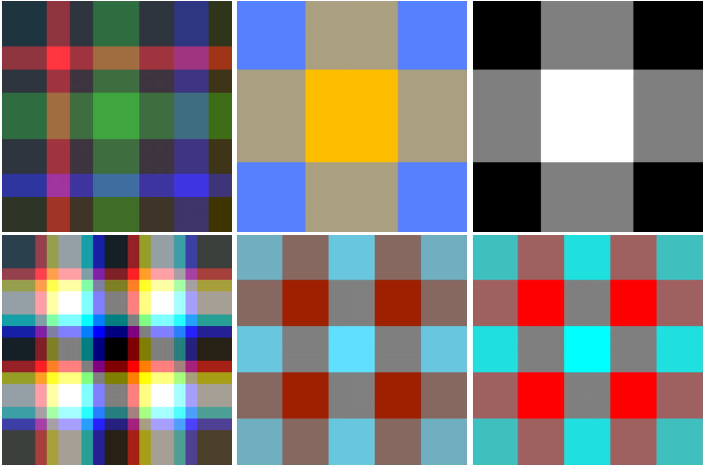
The functions whose divergence generates singular vectors reduce to

$$z_k^1(x_1, x_2) = c_k^1 l_k^1(x_1) \text{ and } z_k^2(x_1, x_2) = c_k^2 l_k^2(x_2),$$

where $c_k^r \in \boldsymbol{R}$, $|l_k^r(x)| \leq 1$, l_k^r piecewise linear and linearity changes iff $|l_k^r(x)| = 1$.



CTV	Singular Vectors	Properties
$\ell^{1,1,1}$	$u_k(x_1, x_2) = -c_k^1 D_1 l_k^1(x_1) - c_k^2 D_2 l_k^2(x_2)$	l_k^r depend on k and $c_k^r \in \{0, \pm 1\}$
$\ell^{2,1,1}$	$u_k(x_1, x_2) = -c_k^1 D_1 l^1(x_1) - c_k^2 D_2 l^2(x_2)$	l^r do not depend on k and $\|c^r\|_2 = 1$
$\ell^{\infty,1,1}$	$u_k(x_1, x_2) = -c_k^1 D_1 l^1(x_1) - c_k^2 D_2 l^2(x_2)$	l^r do not depend on k and $c_k^r \in \{0, \pm 1\}$



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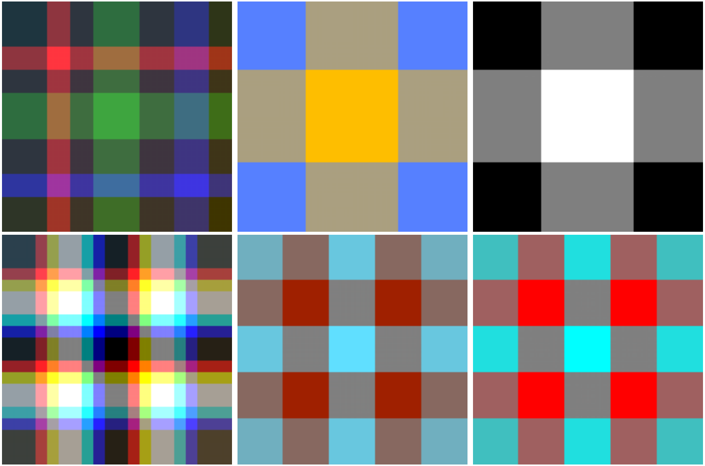

$\ell^{1,1,1}$ $\ell^{2,1,1}$ $\ell^{\infty,1,1}$

The ℓ^∞ norm introduces the strongest channel coupling!

Minimization using the Primal-Dual Algoritm

- Primal formulation:

$$\min_{\mathbf{u}\in R^{N\times C}} F(\mathbf{u}) + G(\mathbf{u}) = \|D\mathbf{u}\|_{\vec{b},a} + G(\mathbf{u}) .$$

Minimization using the Primal-Dual Algoritm

- Primal formulation:

$$\min_{\mathbf{u}\in R^{N\times C}} F(\mathbf{u}) + G(\mathbf{u}) = \|D\mathbf{u}\|_{\vec{b},a} + G(\mathbf{u}).$$

- Since F is closed and l.s.c., then

$$F(D\mathbf{u}) = F^{**}(D\mathbf{u}) = \sup_{\mathbf{p}\in R^{N\times M\times C}} \langle D\mathbf{u}, \mathbf{p}\rangle - F^{*}(\mathbf{p}).$$

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- If $F = \|\cdot\|$, then its Legendre-Fenchel transform is the indicator function of the unit dual norm ball:

$$F^{*}(\mathbf{p}) = \left\{ \begin{array}{ll} 0 & \text{if } \|\mathbf{p}\|_{\vec{b}^{*},a*} \leq 1, \\ +\infty & \text{otherwise.} \end{array} \right\} = \mathcal{X}_{\|\cdot\|_{\vec{b}^{*},a*}\leq 1}(\mathbf{p}).$$

Theorem

Let $\|\cdot\|_{\vec{b}^{*}}$ and $\|\cdot\|_{a^{*}}$ be the dual norms to $\|\cdot\|_{\vec{b}}$ and $\|\cdot\|_{a}$, respectively. Consider $A \in \boldsymbol{R}^{N\times M\times C}$ and define $v \in \boldsymbol{R}^{N}$ such that $v_i = \|A_{i,:,:}\|_{\vec{b}^{*}}$ for each $i \in \{1,\ldots,N\}$. If $\|v\|_{a^{*}}$ only depends on the absolute values of v_i's, then the dual norm to $\|\cdot\|_{\vec{b},a}$ is

$$\|A\|_{\vec{b}^{*},a^{*}} = \|v\|_{a^{*}}, \quad \text{with} \quad v_i = \|A_{i,:,:}\|_{\vec{b}^{*}}, \ \forall i \in \{1,\ldots,N\}.$$

- Saddle-point formulation:

$$\min_{\mathbf{u}\in R^{N\times C}} \max_{\mathbf{p}\in R^{N\times M\times C}} \langle D\mathbf{u}, \mathbf{p}\rangle - F^{*}(\mathbf{p}) + G(\mathbf{u}),$$

with optimality conditions

$$0 \in \partial G(\widehat{\mathbf{u}}) + D^{\top}\widehat{\mathbf{p}} \;\text{ and }\; 0 \in \partial F^{*}(\widehat{\mathbf{p}}) - D\widehat{\mathbf{u}}.$$

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- Primal-Dual algorithm [Chambolle, Pock '11]:

$$\begin{aligned}
\mathbf{u}^{n+1} &= \operatorname{prox}_{\tau_n G}\left(\mathbf{u}^n - \tau_n D^\top \mathbf{p}^n\right) && \leftarrow \quad \text{Gradient descent step in } \mathbf{u} \\
\bar{\mathbf{u}}^{n+1} &= \mathbf{u}^{n+1} + \left(\mathbf{u}^{n+1} - \mathbf{u}^n\right), && \leftarrow \quad \text{Over-relaxation step in } \mathbf{u} \\
\mathbf{p}^{n+1} &= \operatorname{prox}_{\sigma_n F^*}\left(\mathbf{p}^n + \sigma_n D\bar{\mathbf{u}}^{n+1}\right) && \leftarrow \quad \text{Gradient ascent step in } \mathbf{p}
\end{aligned}$$

where $\tau_n, \sigma_n > 0$ are adaptive step-size parameters and

$$\operatorname{prox}_{\alpha f}(x) = \arg\min_y \left\{ \frac{1}{2\alpha}\|y - x\|_2^2 + f(y) \right\}.$$

- Saddle-point formulation:

$$\min_{\mathbf{u}\in R^{N\times C}} \max_{\mathbf{p}\in R^{N\times M\times C}} \langle D\mathbf{u}, \mathbf{p}\rangle - F^*(\mathbf{p}) + G(\mathbf{u}),$$

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$$\text{prox}_{\alpha f}(x) = \arg\min_y \left\{ \frac{1}{2\alpha} \|y - x\|_2^2 + f(y) \right\}.$$

- The proximity operator of $F^* = \mathcal{X}_{\|\cdot\|_{\vec{b}^*, a*} \leq 1}$ is the projection onto the unit dual norm ball

$$\widetilde{\mathbf{p}} = \text{prox}_{\sigma F^*}(\mathbf{p}) = \text{proj}_{\|\cdot\|_{\vec{b}^*, a*} \leq 1}(\mathbf{p}).$$

Experimental Results
Image denoising

Noisy ($\sigma = 30$)

$\ell^{1,1,1}(col, der, pix)$

Experimental Results
Image denoising

Noisy ($\sigma = 30$)

$\left(S^{\infty}(col, der), \ell^{1}(pix)\right)$

Experimental Results
Image denoising

Noisy ($\sigma = 30$)

$\ell^{2,1,1}(col, der, pix)$

Experimental Results
Image denoising

Noisy ($\sigma = 30$)

$\ell^{2,\infty,1}(der, col, pix)$

Experimental Results

Image denoising

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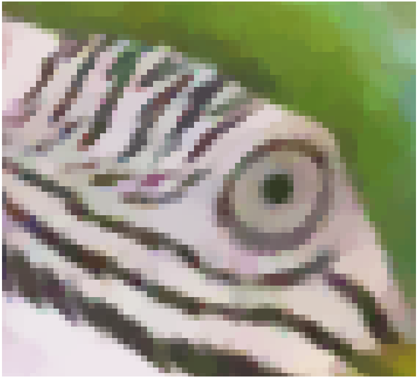


$(S^1(col, der), \ell^1(pix))$

Experimental Results
Image denoising

Noisy ($\sigma = 30$)

$\ell^{\infty,1,1}(col, der, pix)$

Behaviour of CTV methods w.r.t. changing regularization parameter

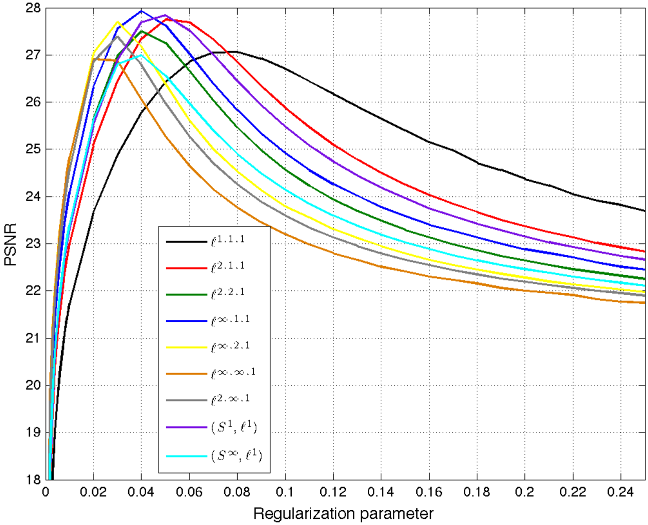


Image denoising on Kodak dataset

	Noisy	$\ell^{1,1,1}$	$\ell^{2,1,1}$	$\ell^{2,2,1}$	$\ell^{\infty,1,1}$	$\ell^{\infty,2,1}$	$\ell^{\infty,\infty,1}$	$\ell^{2,\infty,1}$	S^1, ℓ^1	S^∞, ℓ^1
1	24.78	28.14	29.07	28.51	**29.90**	29.19	28.60	29.07	29.20	27.96
2	24.76	28.54	29.48	29.22	**30.18**	29.87	29.36	29.66	29.83	28.62
3	24.80	29.20	30.15	29.81	**30.85**	30.51	29.84	30.25	30.33	29.24
4	24.68	30.92	32.22	31.80	**32.73**	32.71	31.54	32.13	32.32	31.01
5	24.71	31.50	32.75	32.41	33.13	**33.30**	32.10	32.64	32.81	31.65
6	24.72	27.36	28.19	27.98	**29.01**	28.64	28.29	28.52	28.59	27.47
7	24.71	29.46	30.39	30.12	**30.86**	30.71	29.99	30.35	30.57	29.53
8	24.96	31.08	32.10	31.84	**32.41**	32.40	31.62	32.02	32.20	31.22
9	25.68	30.92	31.74	31.54	**32.10**	32.00	31.49	31.78	31.85	31.11
10	24.66	29.75	30.81	30.49	**31.48**	31.29	30.52	30.94	31.05	29.84
11	24.66	30.14	31.10	30.84	**31.49**	31.46	30.68	31.07	31.22	30.25
12	24.71	31.85	33.15	32.84	33.45	**33.69**	32.47	33.03	33.25	32.05
	24.82	29.91	30.93	30.62	**31.47**	31.31	30.54	30.96	31.10	30.00

Image denoising on BSDS dataset

	Noisy	$\ell^{1,1,1}$	$\ell^{2,1,1}$	$\ell^{2,2,1}$	$\ell^{\infty,1,1}$	$\ell^{\infty,2,1}$	$\ell^{\infty,\infty,1}$	$\ell^{2,\infty,1}$	(S^1,ℓ^1)	(S^∞,ℓ^1)
1	24.88	29.70	30.56	30.41	**30.82**	30.72	30.17	30.46	30.80	29.85
2	25.02	30.01	30.98	30.52	**31.54**	31.03	30.44	30.87	31.12	29.95
3	25.04	30.26	31.03	30.86	**31.43**	31.26	30.78	31.04	31.24	30.41
4	24.96	32.59	33.73	33.66	33.99	**34.01**	33.36	33.72	34.00	32.93
5	24.72	30.16	30.88	30.75	**31.18**	31.07	30.62	30.87	31.01	30.32
6	25.03	29.24	30.19	29.77	**30.89**	30.36	29.84	30.22	30.37	29.27
7	24.65	29.12	30.11	29.74	**30.88**	30.44	29.81	30.22	30.32	29.15
8	24.71	30.57	31.62	31.51	**32.11**	32.09	31.44	31.75	31.92	30.82
9	24.70	31.05	31.94	31.75	**32.11**	32.01	31.32	31.69	32.04	31.20
10	25.42	31.19	31.93	31.87	31.90	31.86	31.34	31.57	**32.10**	31.37
11	24.72	28.06	29.06	28.92	**30.02**	29.69	29.48	29.60	29.64	28.36
12	24.64	30.82	31.86	31.58	**32.19**	31.97	31.20	31.67	32.03	30.89
	24.87	30.23	31.16	30.95	**31.59**	31.38	30.82	31.14	31.38	30.38

Image denoising on McMaster dataset

	Noisy	$\ell^{1,1,1}$	$\ell^{2,1,1}$	$\ell^{2,2,1}$	$\ell^{\infty,1,1}$	$\ell^{\infty,2,1}$	$\ell^{\infty,\infty,1}$	$\ell^{2,\infty,1}$	S^1, ℓ^1	S^∞, ℓ^1
1	25.32	29.29	29.83	29.64	29.74	29.52	28.97	29.25	**29.98**	29.16
2	24.90	27.80	28.41	28.26	28.43	28.32	27.80	28.02	**28.60**	27.75
3	25.46	30.44	30.96	30.84	30.78	30.66	30.16	30.39	**31.17**	30.33
4	25.14	29.26	29.91	29.75	29.95	29.82	29.30	29.54	**30.13**	29.22
5	25.62	31.11	31.46	31.40	30.97	30.84	30.33	30.55	**31.64**	30.89
6	25.01	29.83	30.49	30.32	30.34	30.13	29.55	29.84	**30.74**	29.68
7	25.21	30.96	31.63	31.48	31.41	31.21	30.66	30.98	**31.80**	30.87
8	25.34	31.98	32.72	32.60	32.50	32.30	31.78	32.15	**32.88**	31.99
9	25.21	32.54	33.36	33.32	33.08	32.93	32.50	32.85	**33.53**	32.70
10	24.69	32.26	33.06	33.02	32.70	32.54	32.10	32.49	**33.20**	32.37
11	25.55	30.21	30.85	30.75	30.87	30.73	30.35	30.59	**30.98**	30.29
12	25.21	30.58	31.18	30.99	31.11	30.87	30.36	30.69	**31.30**	30.50
	25.22	30.52	31.16	31.03	30.99	30.82	30.32	30.61	**31.33**	30.48

Image denoising on ARRI dataset

	Noisy	$\ell^{1,1,1}$	$\ell^{2,1,1}$	$\ell^{2,2,1}$	$\ell^{\infty,1,1}$	$\ell^{\infty,2,1}$	$\ell^{\infty,\infty,1}$	$\ell^{2,\infty,1}$	(S^1, ℓ^1)	(S^∞, ℓ^1)
1	24.85	30.93	31.79	31.61	**31.94**	31.79	31.29	31.63	31.90	31.03
2	24.79	33.26	34.42	34.06	**34.55**	34.40	33.54	34.08	34.43	33.33
3	24.84	33.89	34.81	34.83	34.94	**35.23**	34.69	34.89	34.99	34.33
4	25.23	34.61	35.40	35.40	35.43	**35.59**	35.16	35.37	35.49	35.01
5	24.66	33.43	34.18	34.12	34.13	**34.24**	33.76	34.04	34.20	33.70
6	24.74	29.33	30.23	30.11	30.35	30.27	29.80	30.10	**30.46**	29.53
7	25.21	33.17	**33.81**	33.68	33.56	33.50	32.95	33.28	**33.81**	33.25
8	24.65	31.30	32.19	31.82	**32.36**	31.95	31.25	31.75	32.18	31.18
	24.87	32.49	33.35	33.20	33.41	33.37	32.81	33.14	**33.43**	32.67

Experimental Results

Image denoising: local vs nonlocal CTV

Noisy

$\ell^{1,1,1} - \text{TV}$, PSNR $= 33.60$

Experimental Results

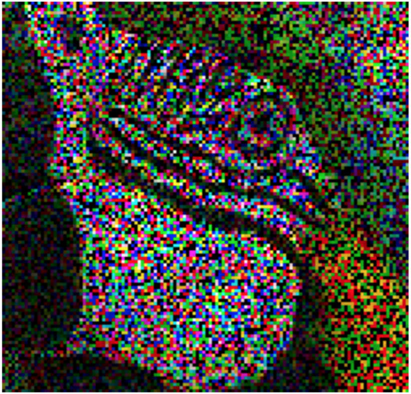
Image denoising: local vs nonlocal CTV

Noisy

$\ell^{1,1,1}$ − NLTV, PSNR = 35.41

Experimental Results

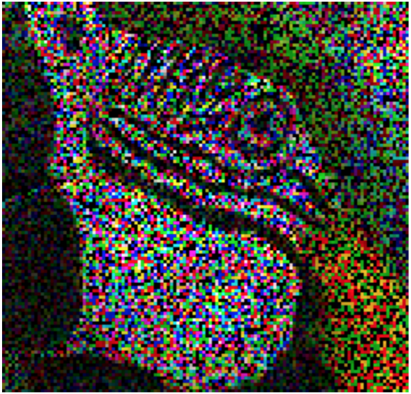
Image inpainting



Noisy ($\sigma = 30$)

$\ell^{1,1,1}(col, der, pix)$

Experimental Results
Image inpainting

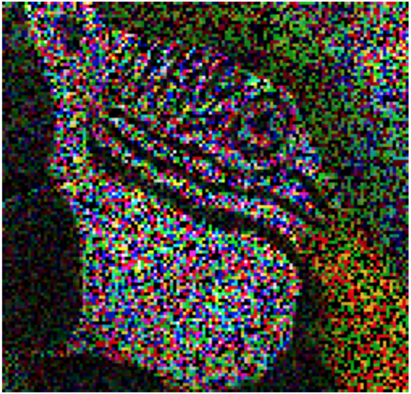


Noisy ($\sigma = 30$)

$(S^{\infty}(col, der), \ell^{1}(pix))$

Experimental Results

Image inpainting

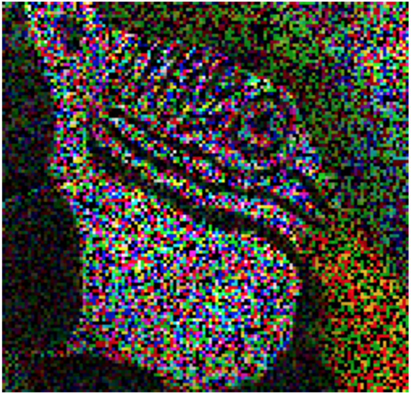


Noisy ($\sigma = 30$)

$\ell^{2,\infty,1}(der, col, pix)$

Experimental Results

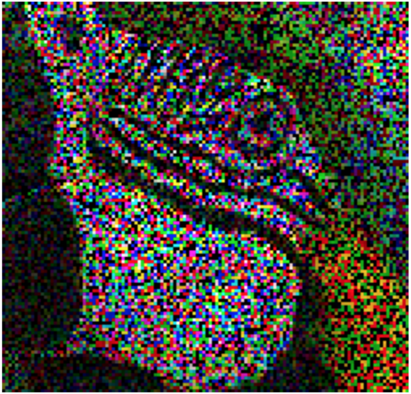
Image inpainting



Noisy ($\sigma = 30$)

$\ell^{2,1,1}(col, der, pix)$

Experimental Results
Image inpainting

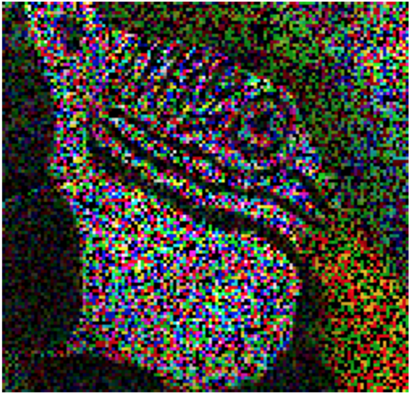


Noisy ($\sigma = 30$)

$\left(S^1(col, der), \ell^1(pix)\right)$

Experimental Results

Image inpainting

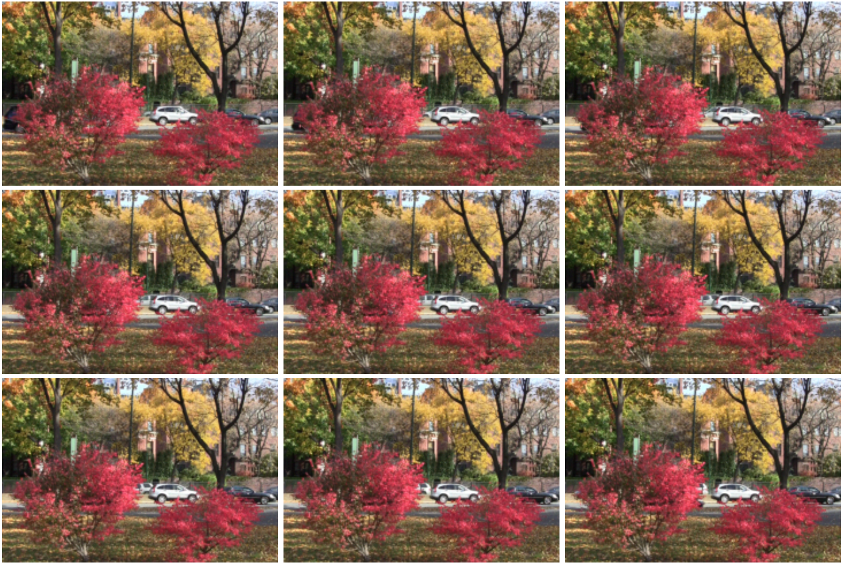


Noisy ($\sigma = 30$)

$\ell^{\infty,1,1}(col, der, pix)$

Future Prospects

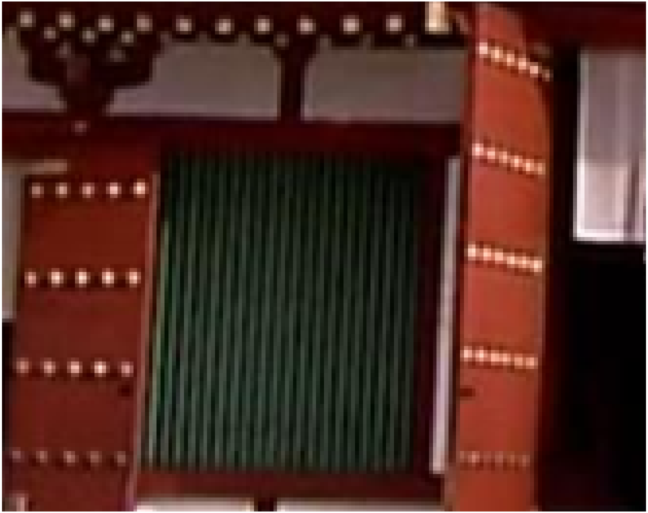
Video super-resolution



Bicubic interpolation

Upsampled stage

Upsampled + deblurred stage



Reference

Bicubic interpolation

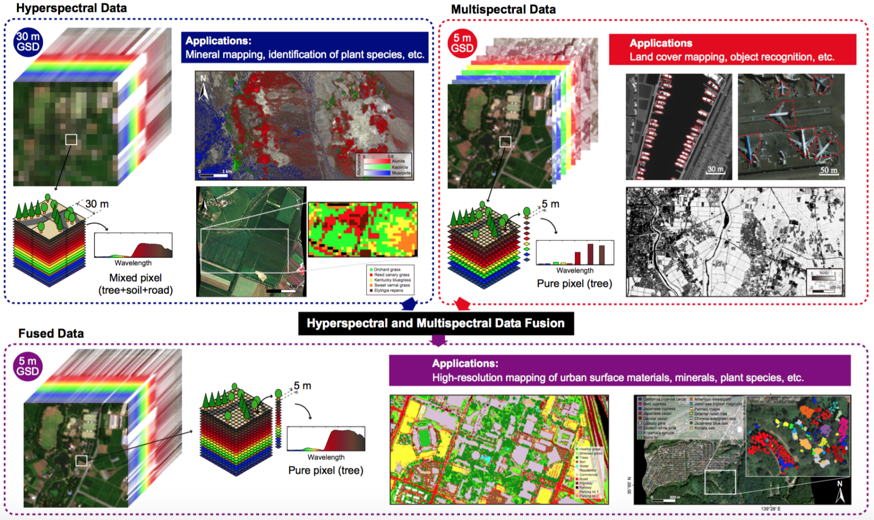
Dong et al. '16

Unger et al. '11

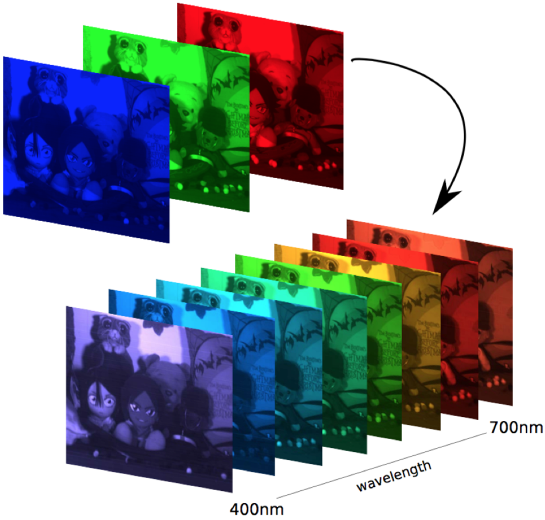
Liao et al. '15

Ours

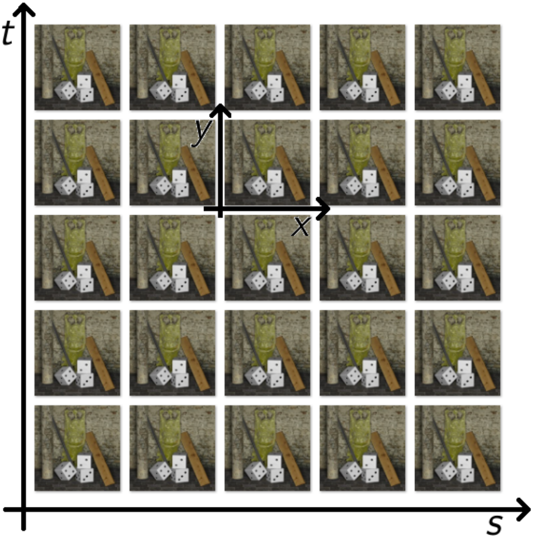
Future Prospects
Hyperspectral data fusion

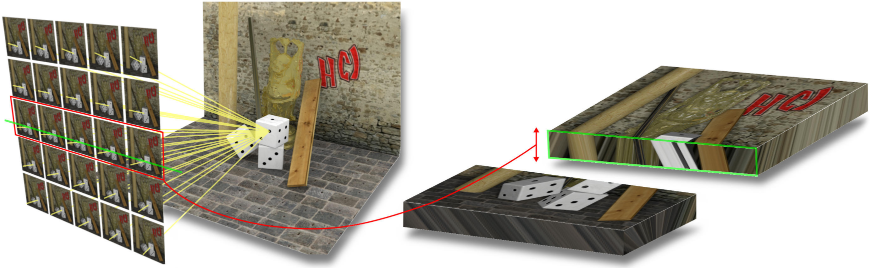


Future Prospects
Spectral super-resolution

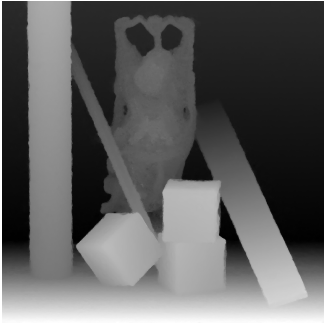


Future Prospects
Light-field imaging





Image



Estimated depth

Conclusions

- We introduced a unified framework for Vectorial Total Variation based on the collaborative enforcing norms $\ell^{p,q,r}$ and (S^p, ℓ^q).
- Depending on the amount of inter-channel correlation, different collaborative norms are suited.
- $\ell^{\infty,1,1}$ and $\left(S^1, \ell^1\right)$ best exploit inter-channel correlations.
- We proposed respective Nonlocal Collaborative TV.
- We proposed the primal-dual algorithm to solve the minimization problem.

Conclusions

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References

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- D., Moeller, Sbert, Cremers, "On the Implementation of Collaborative TV Regularization: Application to Cartoon + Texture Decomposition", IPOL, 2016.
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Collaborative Regularization Approaches in Multi-Channel Variational Imaging

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Proximity operator of $\ell^{p,q,r}$ norms

- $\ell^{1,1,1}-$norm:

$$\left(\mathrm{prox}_{\frac{1}{\sigma}\|\cdot\|_{1,1,1}}(A)\right)_{i,j,k} = \max\left(|A_{i,j,k}| - \frac{1}{\sigma}, 0\right)\mathrm{sign}(A_{i,j,k}).$$

- $\ell^{2,1,1}-$norm:

$$\left(\mathrm{prox}_{\frac{1}{\sigma}\|\cdot\|_{2,1,1}}(A)\right)_{i,j,k} = \max\left(\|A_{i,j,:}\|_2 - \frac{1}{\sigma}, 0\right)\frac{A_{i,j,k}}{\|A_{i,j,:}\|_2}.$$

- $\ell^{2,2,1}-$norm:

$$\left(\mathrm{prox}_{\frac{1}{\sigma}\|\cdot\|_{2,2,1}}(A)\right)_{i,j,k} = \max\left(\|A_{i,:,:}\|_{2,2} - \frac{1}{\sigma}, 0\right)\frac{A_{i,j,k}}{\|A_{i,:,:}\|_{2,2}}.$$

- $\ell^{\infty,1,1}-$norm decouples at each j and k so we are left with an ℓ^{∞} problem computed by means of the projection onto unit ℓ^{1} dual ball:

$$\left(\mathrm{prox}_{\frac{1}{\sigma}\|\cdot\|_{\infty,1,1}}(A)\right)_{i,j,k} = A_{i,j,k} - \frac{1}{\sigma}\mathrm{sign}(A_{i,j,k})\left(\mathrm{proj}_{\|\cdot\|_{1}\leq 1}(\sigma|A_{i,j,:}|)\right)_{i,j,k},$$

where $A_{i,j,:}$ denotes the vector obtained by staking third dimension.

- $\ell^{\infty,\infty,1}-$norm:

$$\left(\mathrm{prox}_{\frac{1}{\sigma}\|\cdot\|_{\infty,\infty,1}}(A)\right)_{i,j,k} = A_{i,j,k} - \frac{1}{\sigma}\mathrm{sign}(A_{i,j,k})\left(\mathrm{proj}_{\|\cdot\|_{1,1}\leq 1}(\sigma|A_{i,:,:}|)\right)_{i,j,k},$$

with $A_{i,:,:}$ being the vector obtained by stacking second and third dimensions.

- $\ell^{\infty,2,1}-$norm:

$$\left(\text{prox}_{\frac{1}{\sigma}\|\cdot\|_{\infty,2,1}}(A)\right)_{i,j,k} = A_{i,j,k} - \frac{1}{\sigma}\text{sign}\left(A_{i,j,k}\right)\left(\text{proj}_{\|\cdot\|_{1,2}\leq 1}\left(\sigma|A_{i,:,:}|\right)\right)_{i,j,k},$$

where $\text{proj}_{\|\cdot\|_{1,2}\leq 1}$ denotes the projection onto unit $\ell^{1,2}-$norm ball.

- $\ell^{2,\infty,1}-$norm:

$$\left(\text{prox}_{\frac{1}{\sigma}\|\cdot\|_{2,\infty,1}}(A)\right)_{i,j,k} = \frac{A_{i,j,k}}{\|A_{i,j,:}\|_2}\max\left(\|A_{i,j,:}\|_2 - \frac{1}{\sigma}v_{i,j}, 0\right),$$

where $v_{i,j} = \left(\text{prox}_{\|\cdot\|_1\leq 1}\left(\sigma\left(\|A_{i,j,:}\|_2\right)_j\right)\right)_{i,j}$, and $\left(\|A_{i,j,:}\|_2\right)_j$ denotes the vector obtained by stacking $\|A_{i,j,:}\|_2$ for all j.

Theorem

Let $f : \boldsymbol{R}^{n\times m} \to \boldsymbol{R}^n$ be $f_i(u) := \sqrt{\sum_{j=1}^m u_{i,j}^2} = \|u_{i,:}\|_2$, and let $g : \boldsymbol{R}^n \to \boldsymbol{R}$ be proper convex function being nondecreasing in each argument. Then

$$\left(prox_{\tau(g\circ f)}(u)\right)_{i,j} = \frac{u_{i,j}}{\|u_{i,:}\|_2}\ \max\left(\|u_{i,:}\|_2 - \tau v_i, 0\right),$$

where the v_i's are the components of the vector $v \in \boldsymbol{R}^n$ that solves

$$v = \arg\min_{w\in R^n} \frac{1}{2}\left\|w - \frac{1}{\tau}f(u)\right\|^2 + \frac{1}{\tau}g^*(w).$$

Proximity operators of (S^p, ℓ^q) norms

If $q = 1$, the proximity operator decouples at each pixel:

- Define $M \times C$ submatrix $B_i := (A_{i,j,k})_{j=1,\ldots,M;\, k=1,\ldots,C}$.
- Let $B = B_i^T$, we need to solve at each pixel

$$\min_{D \in R^{M \times C}} \frac{1}{2}\|D - B\|_F^2 + \frac{1}{\sigma}\|D\|_{S^p}.$$

- Computing SVD of $B = U\Sigma_0 V^T$ and $\Sigma = U^T DV$, the problem is equivalent to

$$\min_{D \in R^{M \times C}} \frac{1}{2}\|U^T DV - \Sigma_0\|_F^2 + \frac{1}{\sigma}\|U^T DV\|_{S^p} \Leftrightarrow \min_{\Sigma \in R^{r \times r}} \frac{1}{2}\|\Sigma - \Sigma_0\|_F^2 + \frac{1}{\sigma}\|\Sigma\|_{S^p}.$$

- For diagonal matrices $S^p(\Sigma) = \ell^p\big(\mathrm{diag}(\Sigma)\big)$, so that we finally solve

$$\min_{s \in R^r} \frac{1}{2}\|s - s_0\|_2^2 + \frac{1}{\sigma}\|s\|_p,$$

  where $s_0 = \mathrm{diag}(\Sigma_0)$ and $s = \mathrm{diag}(\Sigma)$.
- Only need to compute eigenvalues, Σ_0, and eigenvectors, V, of $B_i^T B_i$.

- Let $\widehat{\Sigma}$ s.t. $\operatorname{diag}(\widehat{\Sigma}) = \arg\min_s \frac{1}{2}\|s - s_0\|_2^2 + \frac{1}{\sigma}\|s\|_p$
- The proximity operator $\widehat{D} = \arg\min_D \frac{1}{2}\|D - B\|_2^F + \frac{1}{\sigma}\|D\|_{S^p}$ is $\widehat{D} = U\widehat{\Sigma}V^T$.
- Due to $B = U\Sigma_0 V^T$, communitation of diagonal matrices, and $\widehat{\Sigma}\Sigma_0\Sigma_0^\dagger = \widehat{\Sigma}$ – since $\widehat{\Sigma}$ has at most as many nonzero diagonal entries as Σ_0 –, one has

$$\begin{aligned} BV = U\Sigma_0 \Rightarrow BV\widehat{\Sigma} = U\Sigma_0\widehat{\Sigma} = U\widehat{\Sigma}\Sigma_0 \\ \Rightarrow BV\widehat{\Sigma}\Sigma_0^\dagger = U\widehat{\Sigma} \Rightarrow BV\widehat{\Sigma}\Sigma_0^\dagger V^T = U\widehat{\Sigma}V^T = \widehat{D}, \end{aligned}$$

where $\Sigma_0^\dagger$ denotes the pseudo-inverse matrix of Σ_0, i.e.

$$\left(\Sigma_0^\dagger\right)_{i,j} = \begin{cases} \dfrac{1}{(\Sigma_0)_{i,i}} & \text{if } i = j \text{ and } (s_0)_{i,i} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Therefore, the proximity operator is

$$\widehat{D} = BV\widehat{\Sigma}{\Sigma^\dagger}_0 V^T,$$

where

  - $\operatorname{diag}(\Sigma_0)$ consists of the square root of the eigenvalues of $B^T B$.
  - $\operatorname{col}(V)$ are the eigenvectors of $B^T B$.

Image Denoising

$$\min_{u \in R^{N\times C}} \|Ku\|_{b,a} + \frac{\lambda}{2}\|u-f\|_F^2,$$

where $f \in \mathbb{R}^{N\times C}$ is the noisy image, $\lambda > 0$ the regularization parameter, and $\|\cdot\|_{b,a}$ denotes either an $\ell^{p,q,r}$ norm or a Schatten (S^p, ℓ^q) norm.

The **proximity operator** of $G(u) = \frac{\lambda}{2}\|u-f\|_F^2$ is

$$\text{prox}_{\tau G}(u) = \arg\min_{v\in X}\left\{\frac{1}{2}\|v-u\|_F^2 + \tau\frac{\lambda}{2}\|v-f\|_F^2\right\} \Leftrightarrow \text{prox}_{\tau G}(u) = \frac{u+\tau\lambda f}{1+\tau\lambda}.$$

Therefore, the solution of $u^{n+1} = \text{prox}_{\tau_n G}\left(u^n - \tau_n K^T z^n\right)$ is given by

$$u^{n+1} = \frac{u^n + \tau_n\left(-K^T z^n + \lambda f\right)}{1+\tau_n\lambda},$$

where $-K^T = \text{div}$ is defined as $\langle -\text{div}\, z, u\rangle_X = \langle z, Ku\rangle_Y$.

Image Deconvolution

$$\min_{u \in R^{N \times C}} \|Ku\|_{b,a} + \frac{\lambda}{2}\|Au - f\|_F^2,$$

with A being the linear operator modelling the convolution of u with a Gaussian kernel.s

The **proximity operator** of $G(u) = \frac{\lambda}{2}\|Au - f\|_F^2$ is

$$\widehat{u} = \arg\min_{v \in X} \left\{ \frac{1}{2}\|v - u\|_F^2 + \tau \frac{\lambda}{2}\|Av - f\|_F^2 \right\} \Leftrightarrow \widehat{u} = (I + \tau\lambda A^* A)^{-1} (u + \tau\lambda A^* f) \, .$$

Computing $(I + \tau\lambda A^* A)^{-1}$ is huge time consuming in the spatial domain. On the contrary, using FFT, the solution can be efficiently computed as

$$\widehat{u} = \mathcal{F}^{-1} \left(\frac{\mathcal{F}(u) + \tau\lambda\mathcal{F}(A)\mathcal{F}(f)}{1 + \tau\lambda\mathcal{F}(A)^2} \right) .$$